Introduction to Search, Propagation, and Checkers

Modern Constraint Programming

(ESSAI'26)

Emir Demirović

What is a combinatorial optimisation problem?

High School Timetabling

Coordinate teachers, rooms, and curriculum requirements subject to constraints and preferences

Demirović, Musliu; "MaxSAT-based large neighborhood search for high school timetabling“; Computers & Operations Research 78 (2017): 172-180.

Automotive Paintshop Scheduling

Car pieces need to be painted

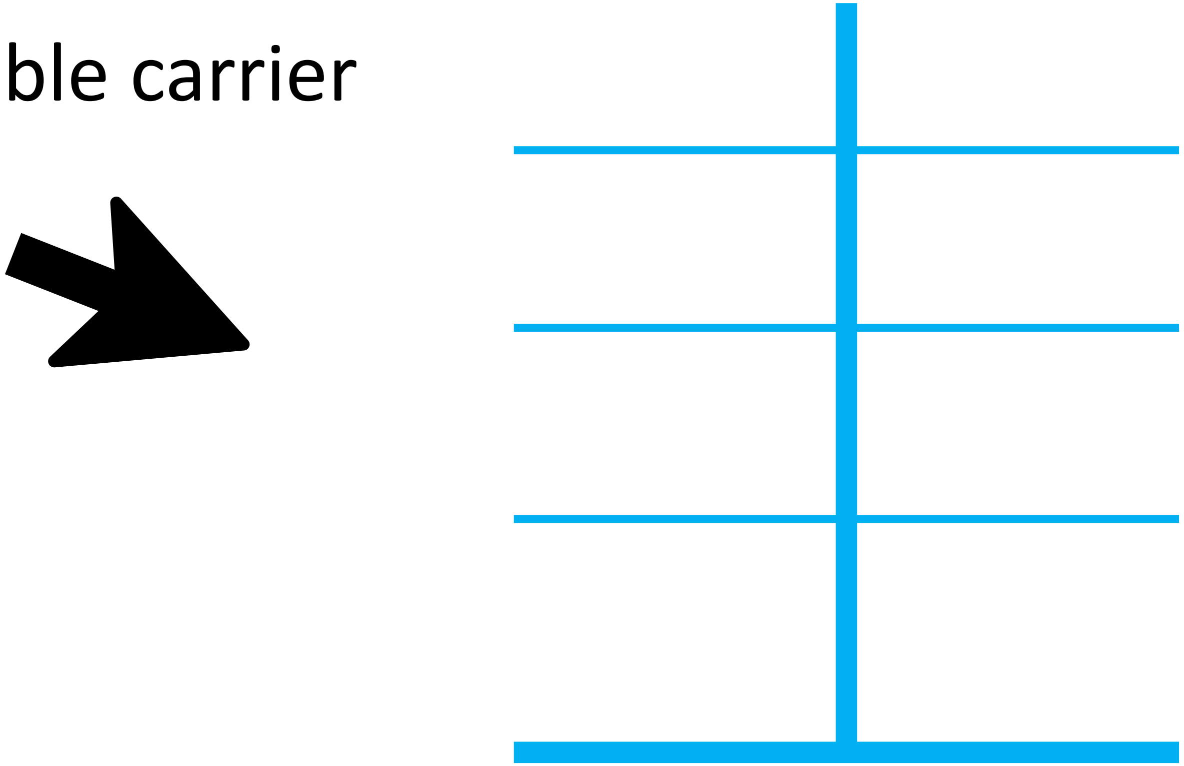
Sophisticated production process

Find the best schedule

Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

Configurable carrier



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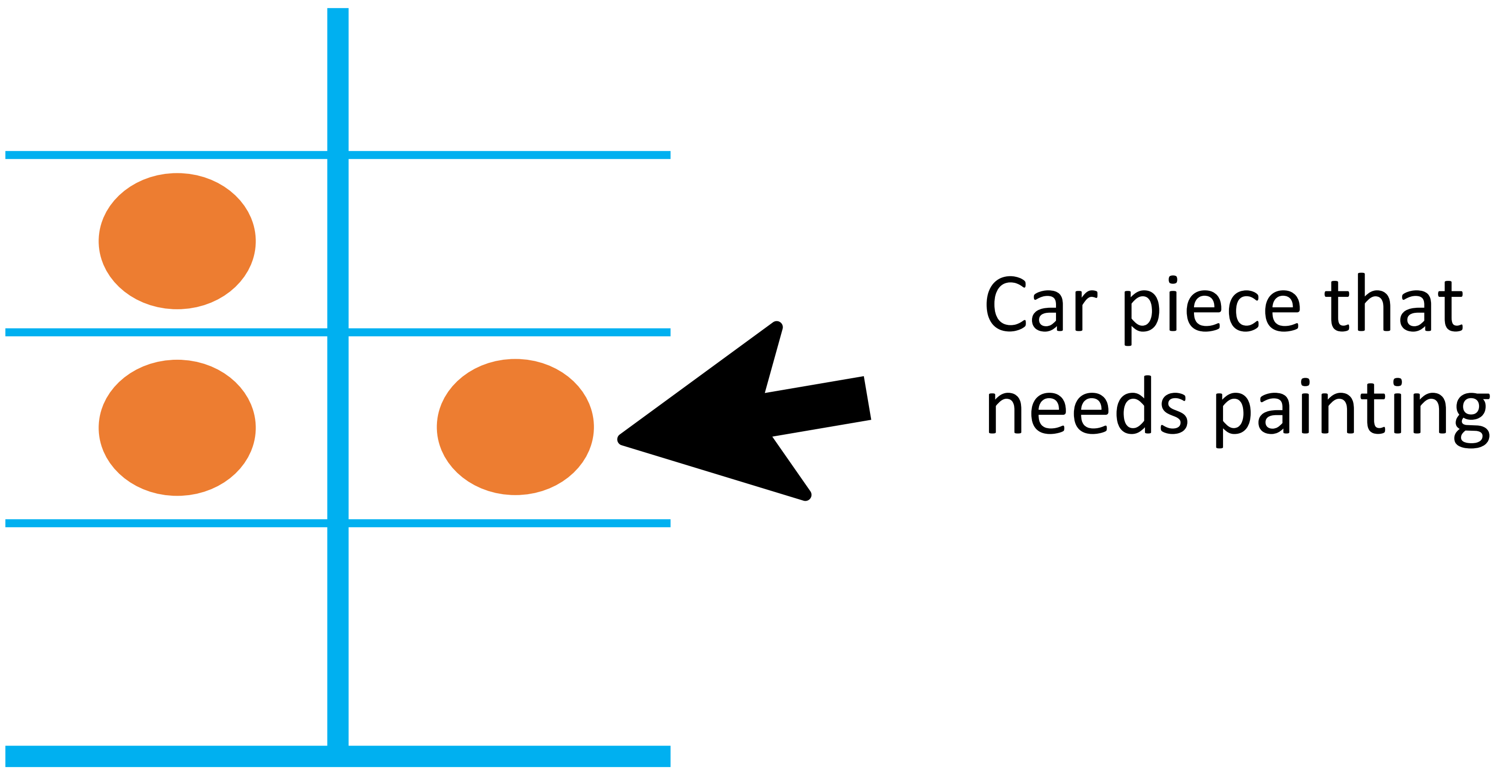
Automotive Paintshop Scheduling



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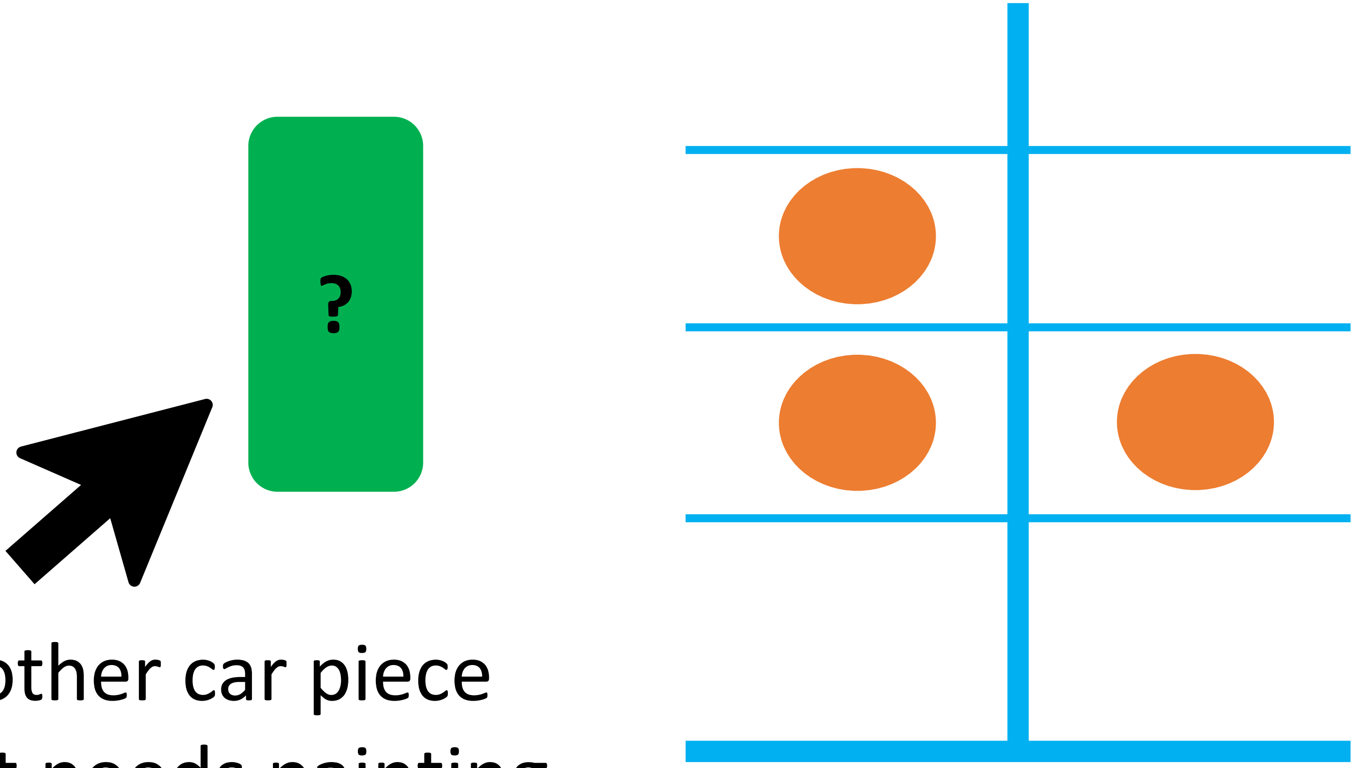
Automotive Paintshop Scheduling

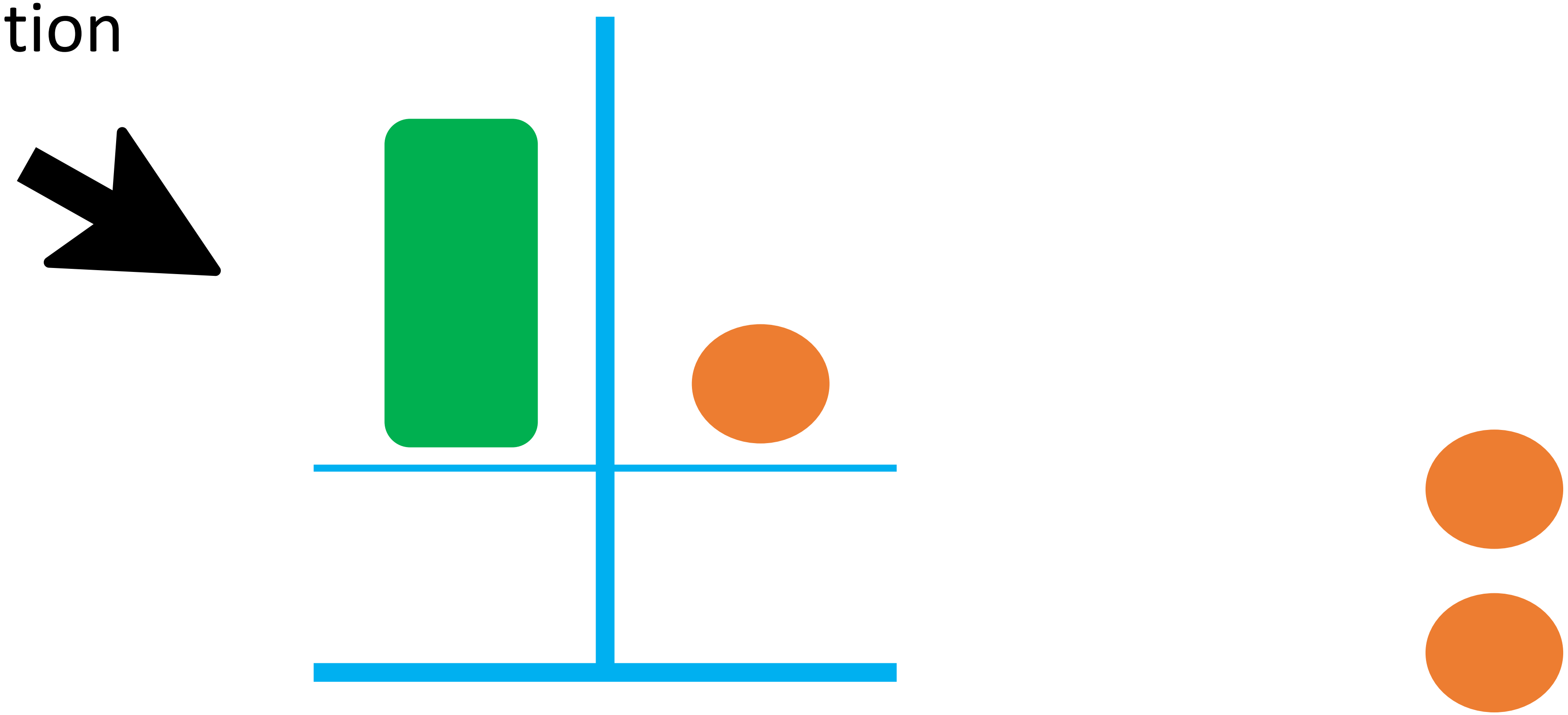


Another car piece
that needs painting

Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

New configuration



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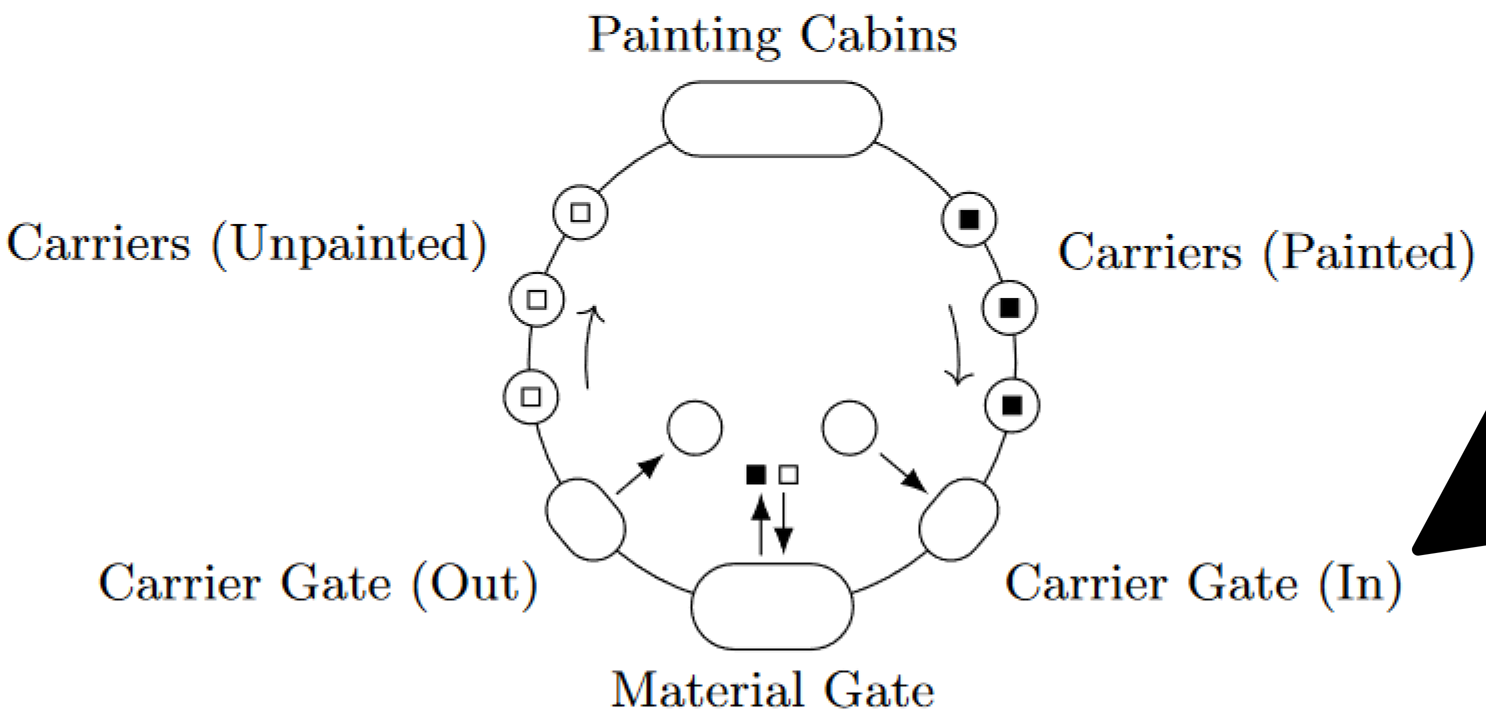
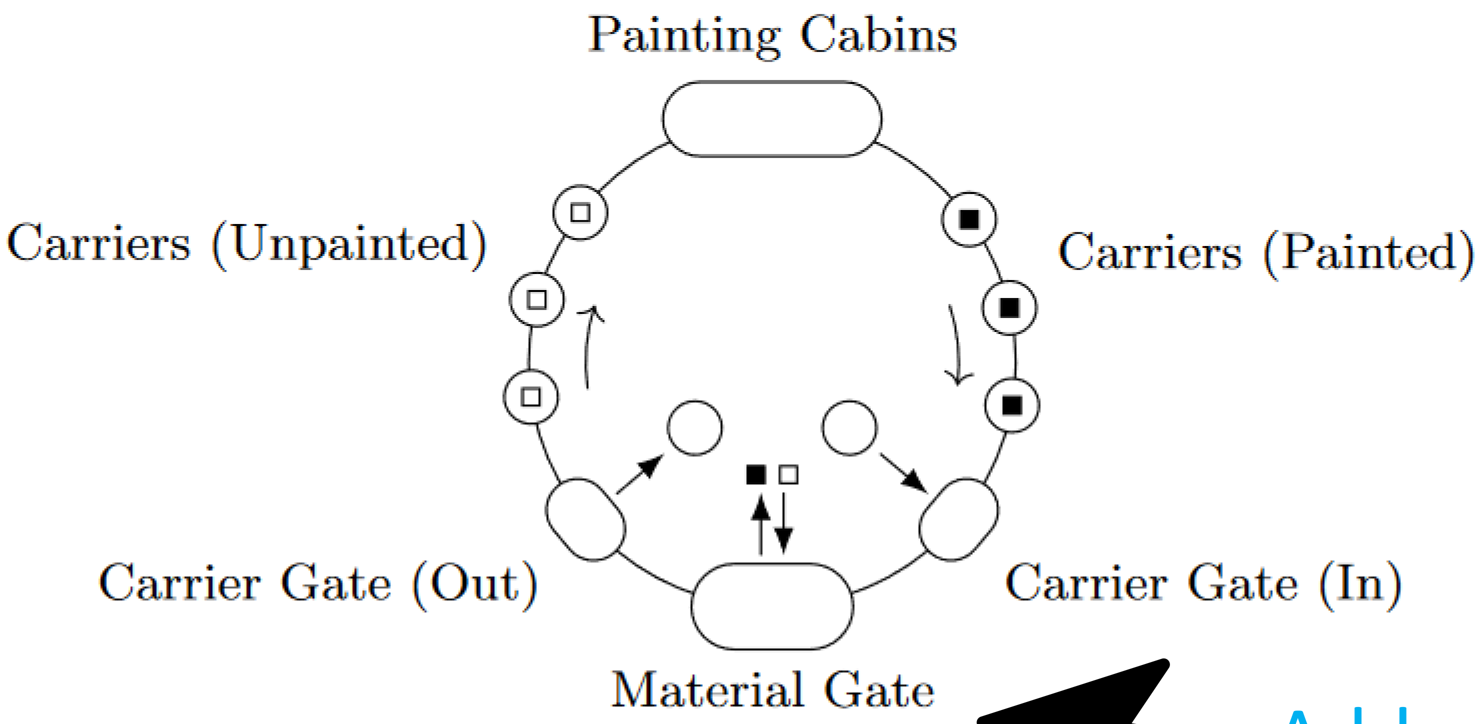
Automotive Paintshop Scheduling



Add new carriers

Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

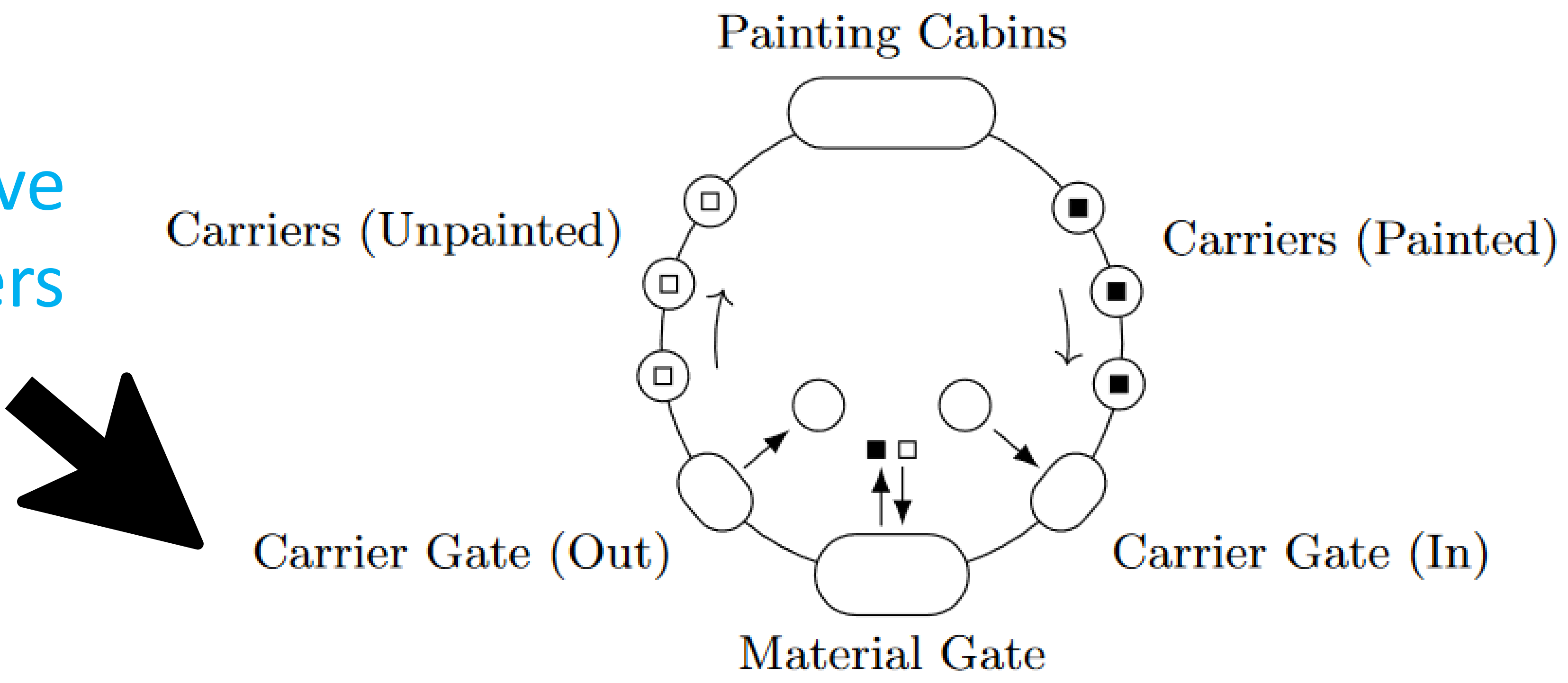


Add pieces on carriers

Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

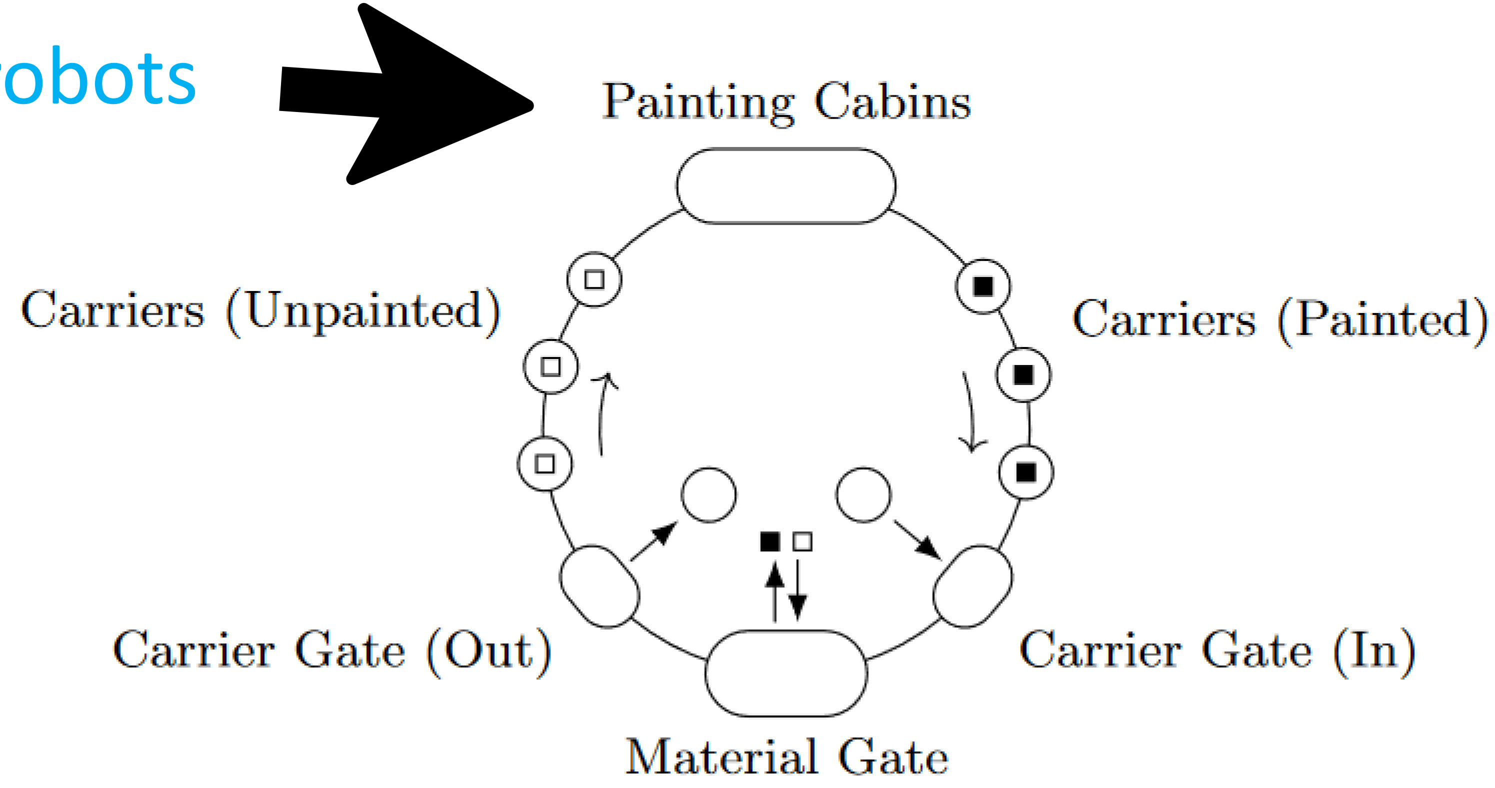
Remove carriers



Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

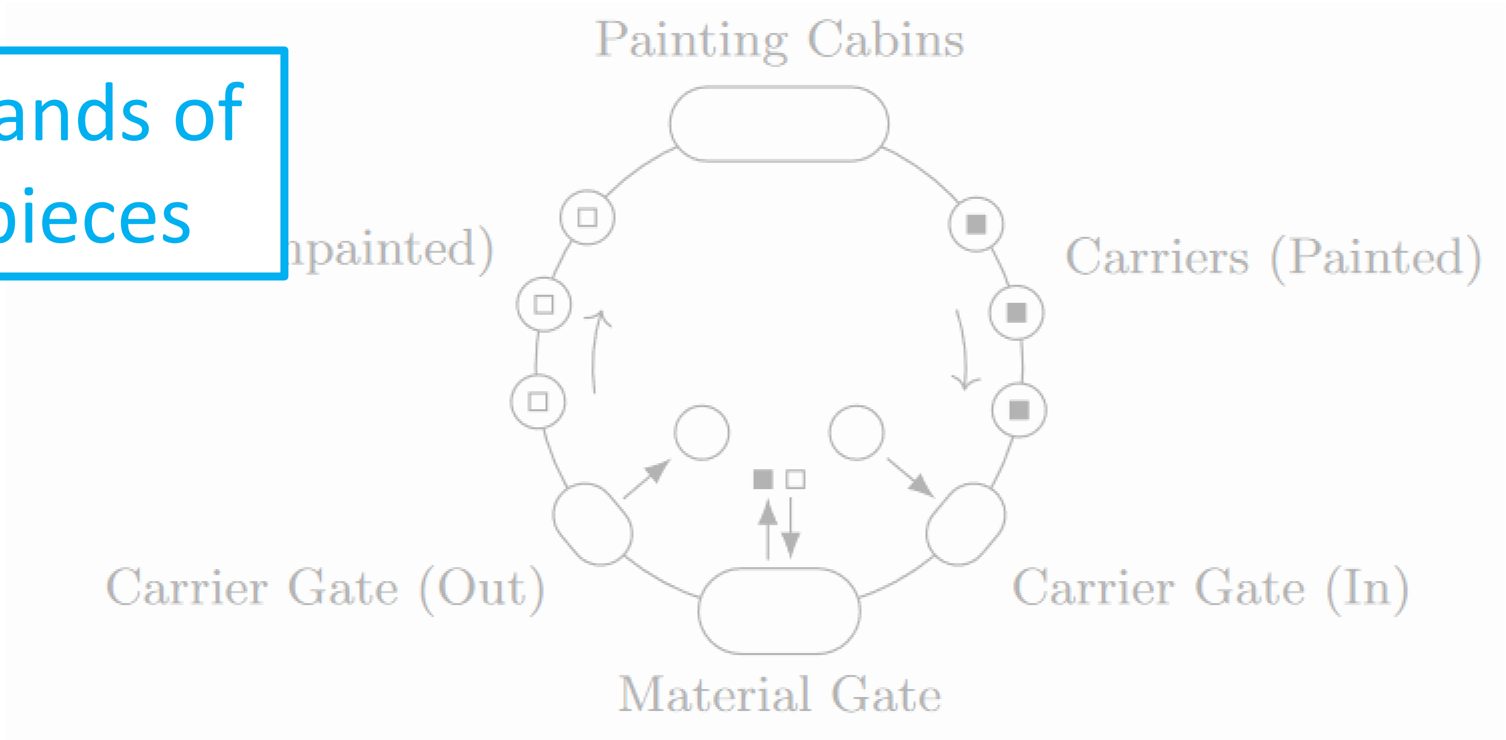
Painting robots



Winter, Musliu, Demirović, Mrkvicka; "Solution approaches for an automotive paint shop scheduling problem." *Proceedings of the International Conference on Automated Planning and Scheduling*. Vol. 29. 2019.

Automotive Paintshop Scheduling

Thousands of car pieces



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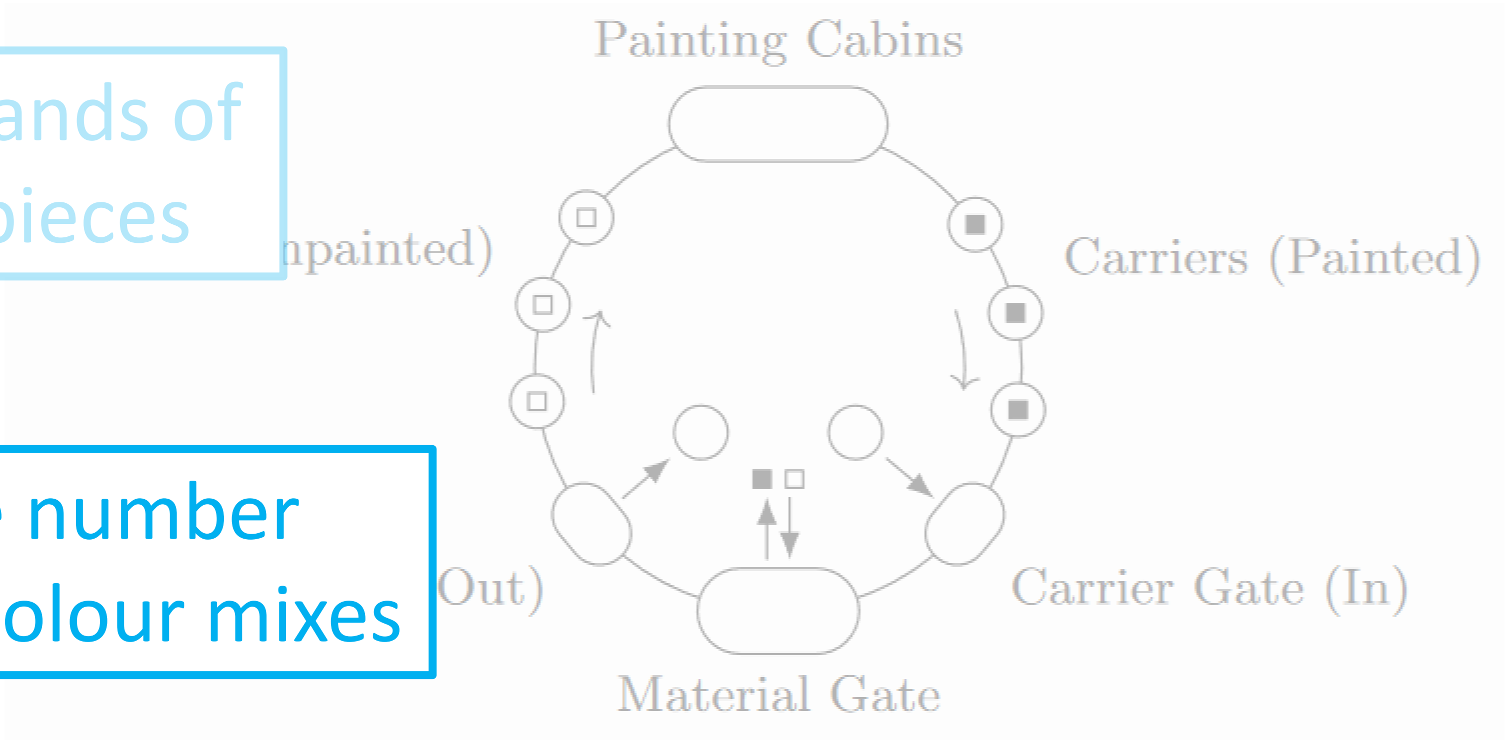
Automotive Paintshop Scheduling

Thousands of car pieces

Incredible number of different colour mixes



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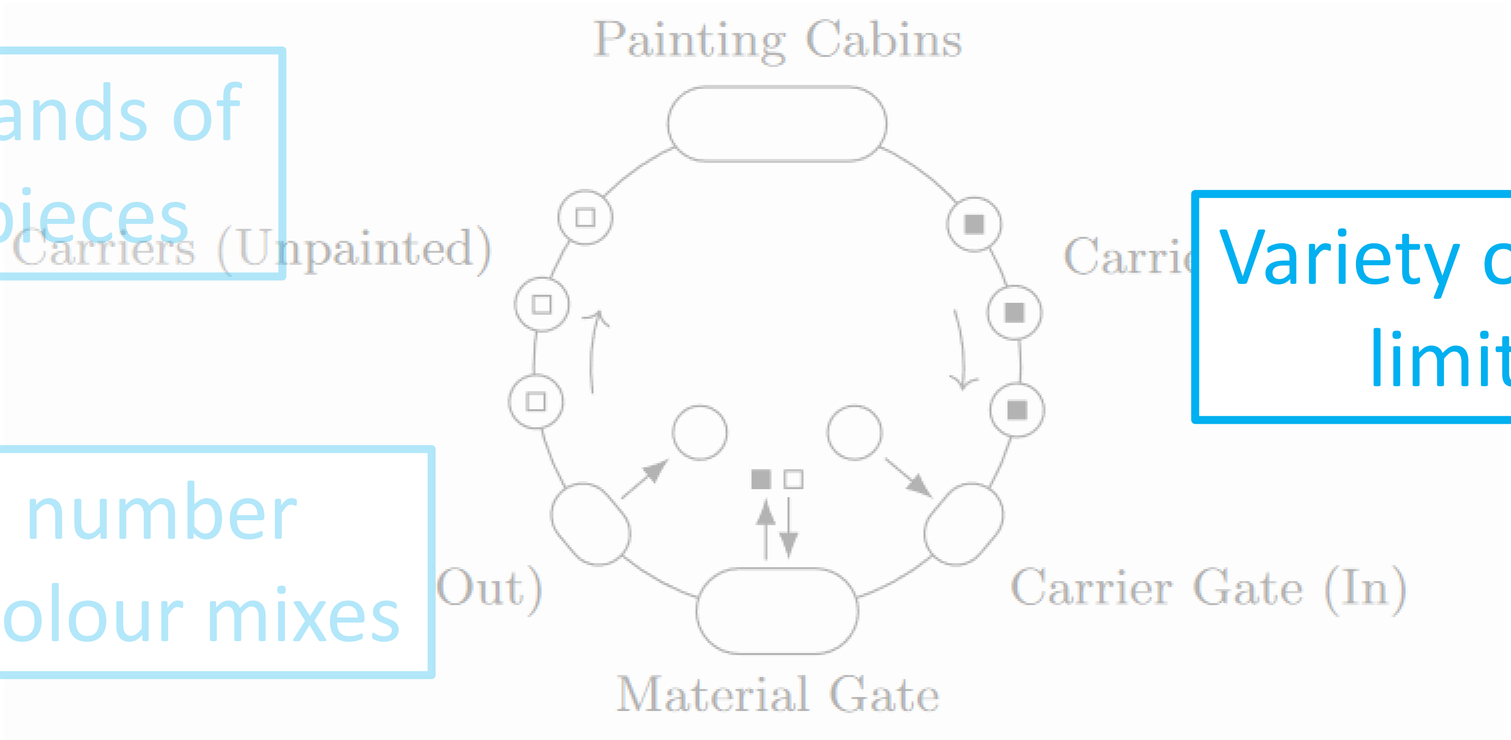
Automotive Paintshop Scheduling

Thousands of car pieces

Incredible number of different colour mixes

Variety of technical limitations



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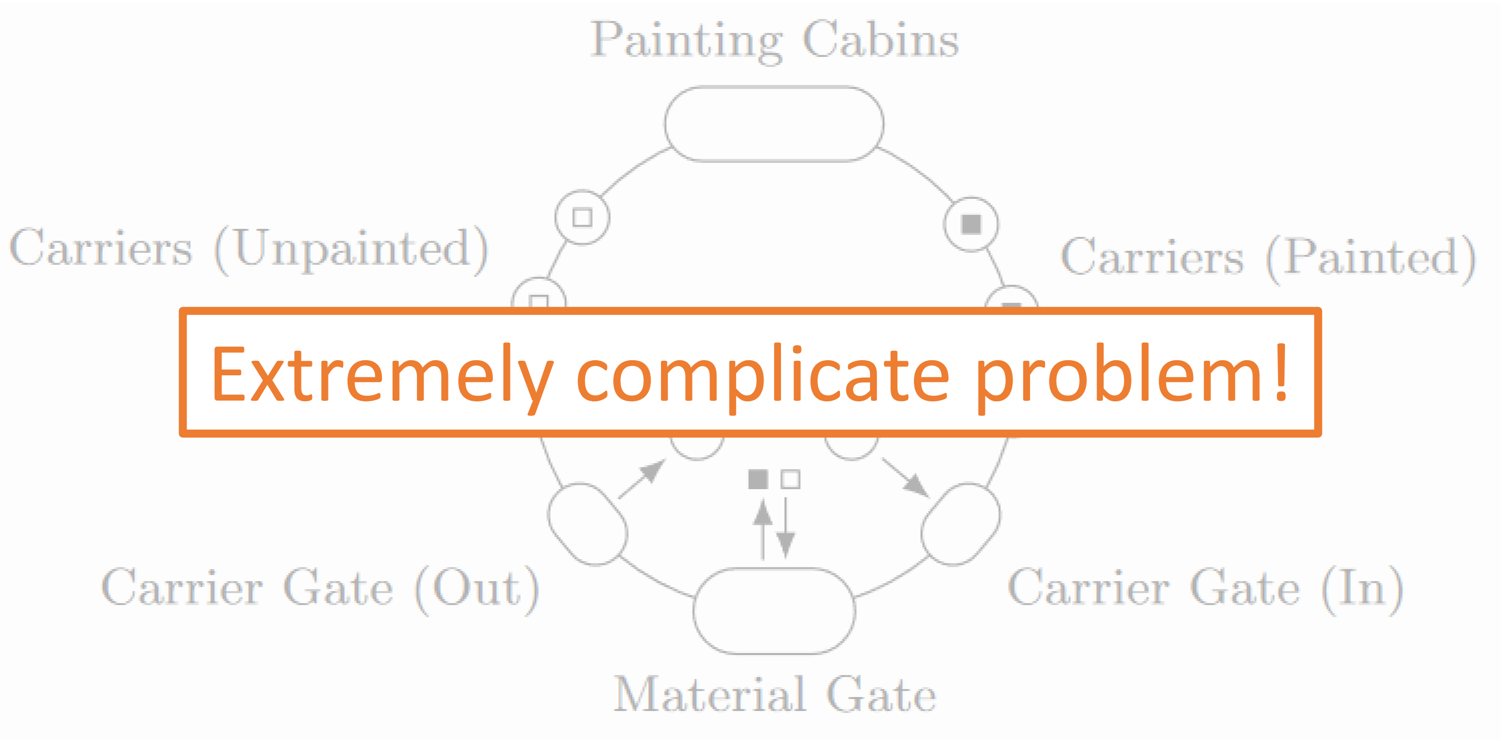
Automotive Paintshop Scheduling



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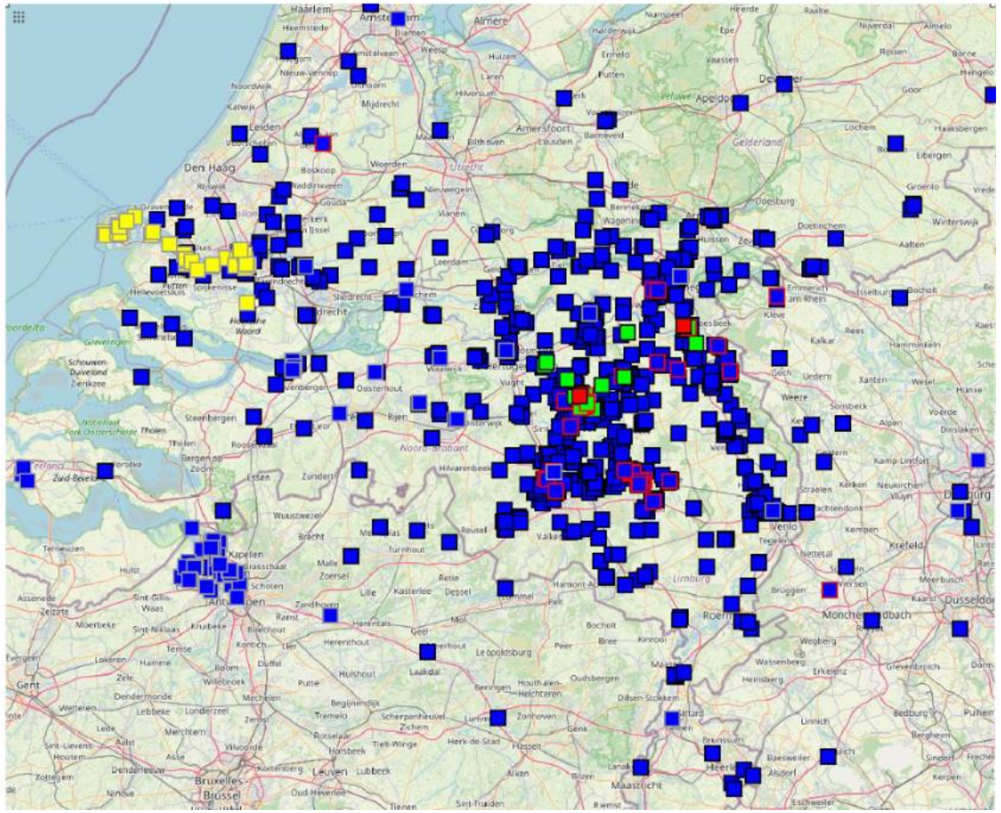
Truck Logistics

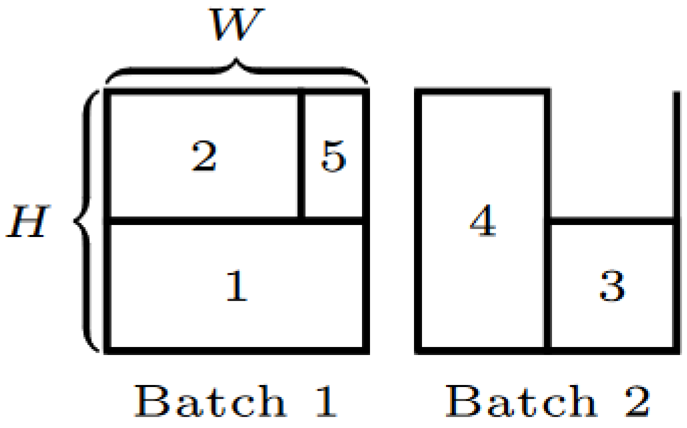


Plan truck routes
to deliver goods

Pingen, Van Ommeren, Van Leeuwen, Fransen, Elfrink, De Vries, Karunakaran, Demirović, Yorke-Smith
"Talking Trucks: Decentralized Collaborative Multi-Agent Order Scheduling for Self-Organizing Logistics„
Proceedings of the International Conference on Automated Planning and Scheduling. Vol. 32. 2022.

Picture taken from the master thesis of Yorick C. de Vries done at TU Delft in 2021

Batch Scheduling for Tool Coating



Tool Coating

Pack tools into batches

Schedule batches into ovens

Horn, Demirović, Yorke-Smith
"Parallel Batch Processing for the Coating Problem„
Proceedings of the International Conference on Automated Planning and Scheduling. 2023.

Fair and Optimal Decision Trees (Machine Learning)

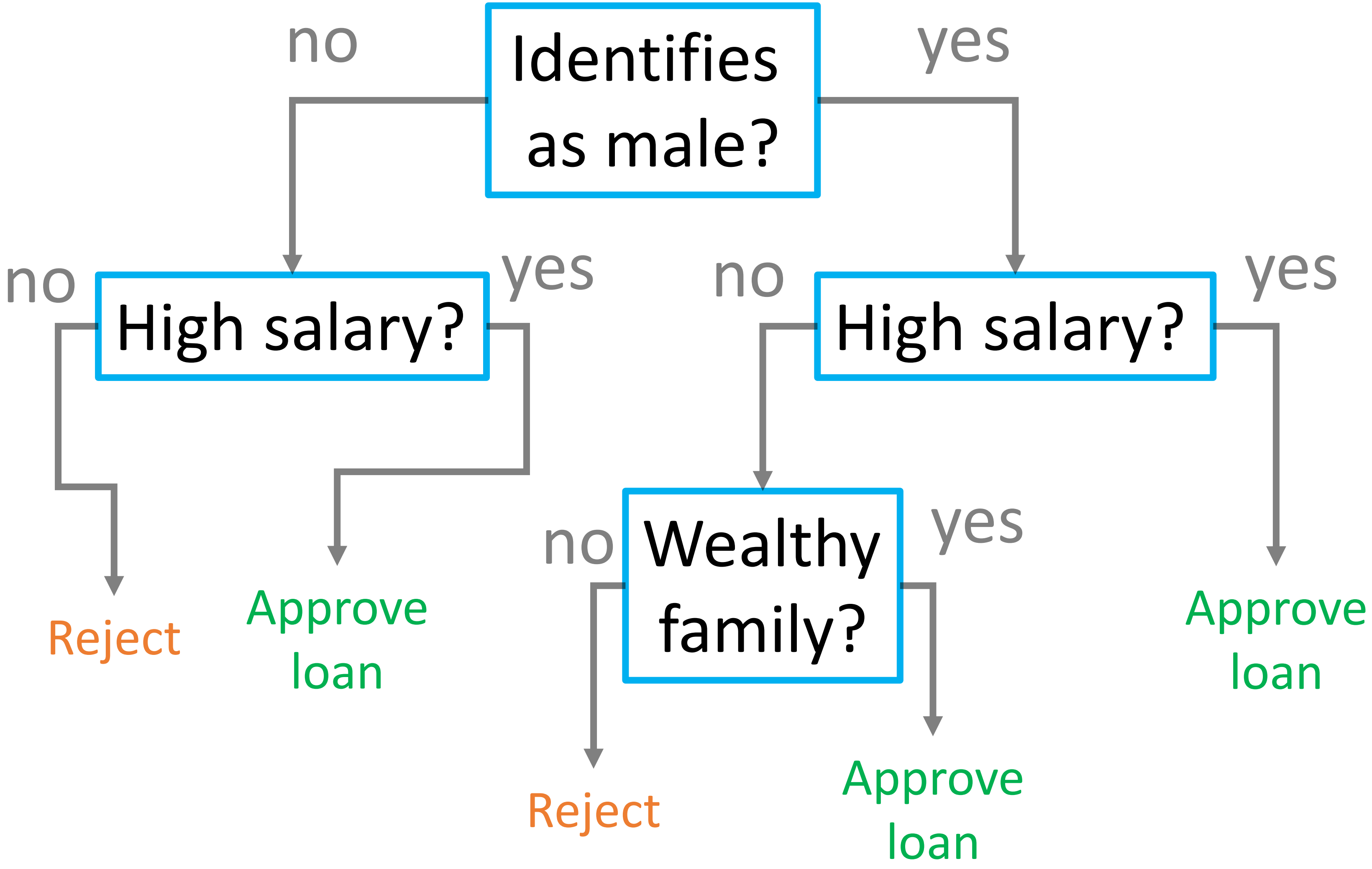
Construct the best
decision tree
based on historical data

"automate the bank loan approval process"

Van Der Linden, De Weerdt, Demirović; "Fair and Optimal Decision Trees: A Dynamic Programming Approach." Advances in Neural Information Processing Systems. 2022.

Fair and Optimal Decision Trees (Machine Learning)

Construct the best decision tree based on historical data



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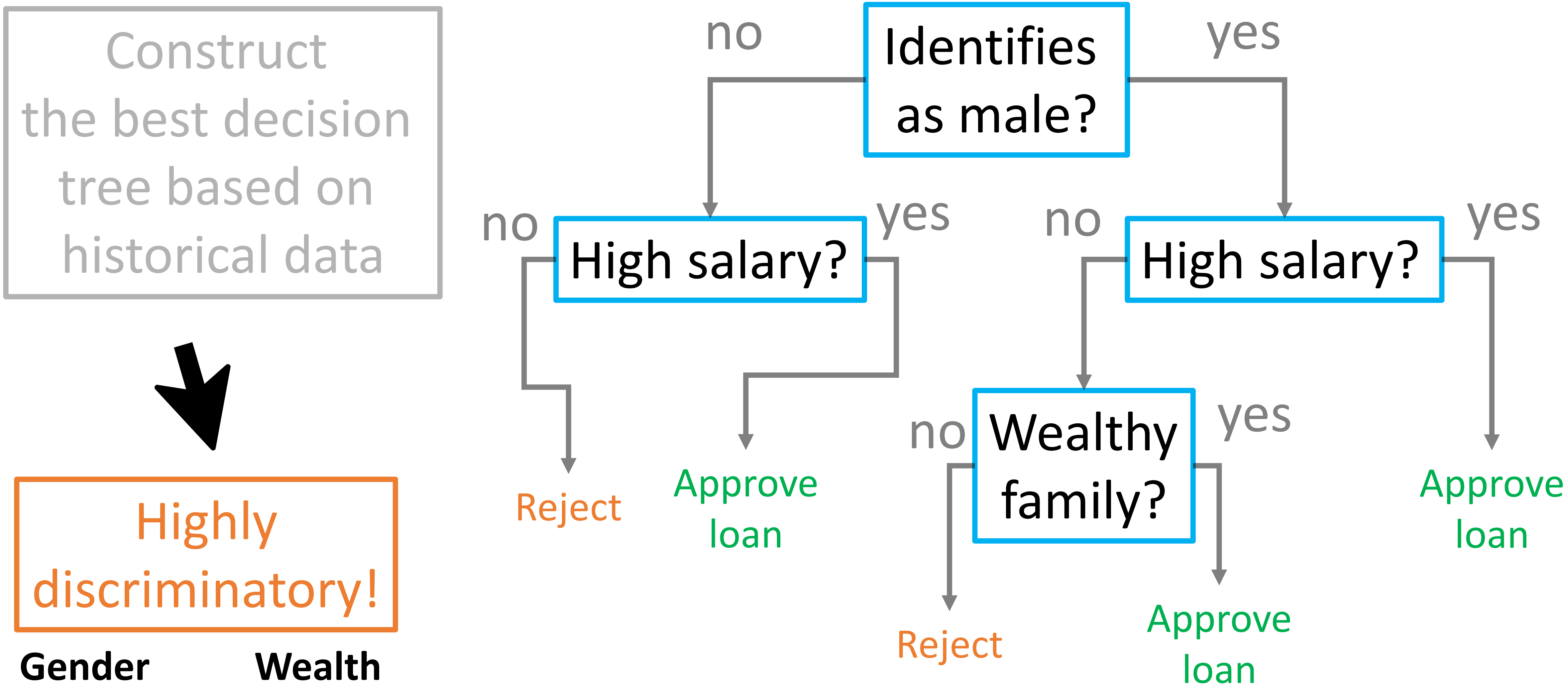

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Fair and Optimal Decision Trees (Machine Learning)

Construct the best
decision tree
based on historical data

Difficult
combinatorial
problem!

...that is also fair and
meaningful for society!

Van Der Linden, De Weerdt, Demirović; "Fair and Optimal Decision Trees: A Dynamic Programming Approach." Advances in Neural Information Processing Systems. 2022.

High School Timetabling

Automotive Paint Shop Scheduling

Truck Logistics

Batch Scheduling for Coating Tools

Fair and Optimal Decision Trees

...all examples of combinatorial optimisation problems!

Combinatorial Optimisation Problem

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Solution

Combinatorial Optimisation Problem

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Set of feasible solutions, implicitly defined through constraints

Combinatorial Optimisation Problem

Objective function

$$\min F(X)$$
$$X \in C \subseteq \mathbb{N}^n$$

Combinatorial Optimisation Problem

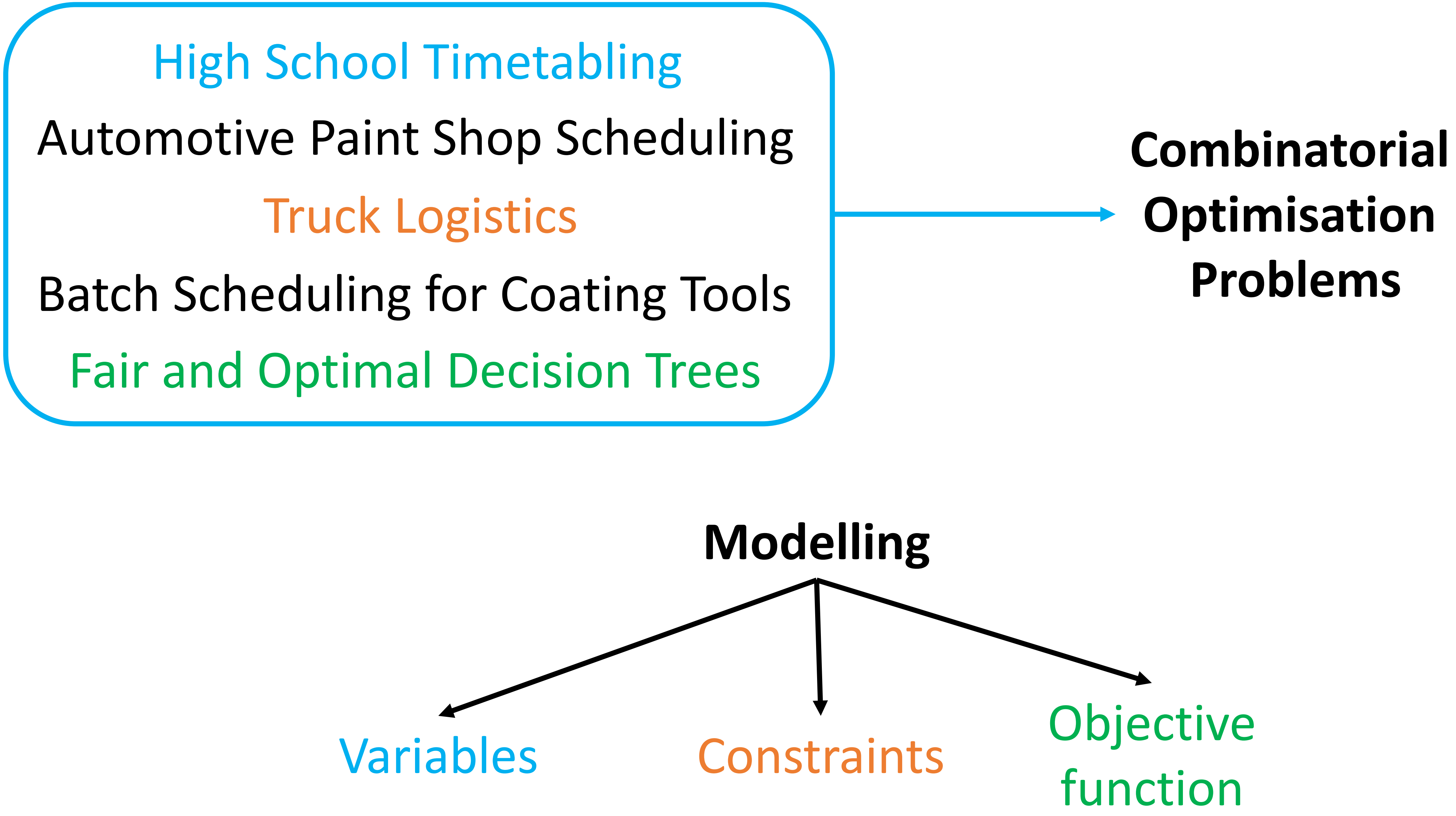
$$\boldsymbol{min\ F(X)}$$
$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Combinatorial Optimisation Problem

$$\min F(X)$$
$$X \in C \subseteq \mathbb{N}^n$$

➔ How to efficiently utilise resources?

Combinatorial optimisation is everywhere!


High School Timetabling
Automotive Paint Shop Scheduling
Truck Logistics
Batch Scheduling for Coating Tools
Fair and Optimal Decision Trees
Combinatorial Optimisation Problems
Modelling
Variables
Constraints
Objective function

Modelling

Variables

decisions
we can make

Constraints

feasibility

Objective
function

solution quality

Modelling

Variables

Constraints

~~Objective function~~

In this course:
Ignore optimisation

decisions
we can make

feasibility

Many optimisation methods
repeatedly call
a satisfaction solver

Modelling

Variables

Constraints

Modelling

Variables

Constraints

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

In this course:

Constraint Programming

Constraints are arbitrary predicates

Constraints

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

$$(\boldsymbol{x} \vee \boldsymbol{y} \vee \boldsymbol{z})$$

$$\sum \boldsymbol{w_i x_i} \leq \boldsymbol{c}$$

Constraints

$$X \in C \subseteq \mathbb{N}^n$$

$$(x \vee y \vee z)$$

Expressive!

$$\sum w_i x_i \leq c$$

All-Different$(x_1, x_2, \ldots, x_n)$

NoOverlap$([x_{11}, x_{12}], [x_{21}, x_{22}], \ldots, [x_{n1}, x_{n2}])$

Constraint Programming

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Day 1: Search, Propagation, Checkers

Day 2: Propagation (All-Different, Cumulative)

Day 3: Conflict Analysis

Day 4: Certification and Proof Systems

Introductory &
Advance Course

Constraint Programming

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Day 1: Search, Propagation, Checkers

Day 2: Propagation (All-Different, Cumulative)

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Introductory & Advance Course

Constraint Programming

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{\boldsymbol{n}}$$

Day 1: Search, Propagation, Checkers

Day 2: Propagation (All-Different, Cumulative)

Day 3: Conflict Analysis

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➘ In class

Intuition

Visuals and Examples

Interaction!

Introductory & Advance Course

Constraint Programming

$$\boldsymbol{X} \in \boldsymbol{C} \subseteq \mathbb{N}^{n}$$

Day 1: Search, Propagation, Checkers

Day 2: Propagation (All-Different, Cumulative)

Day 3: Conflict Analysis

Day 4: Certification and Proof Systems

In class

Intuition

Visuals and Examples

Interaction!

Out of class

Textbook (draft) for details

Today…

Constraint Programming

Search

Constraints and Propagators

Checkers

Problem

↓

Model

Problem

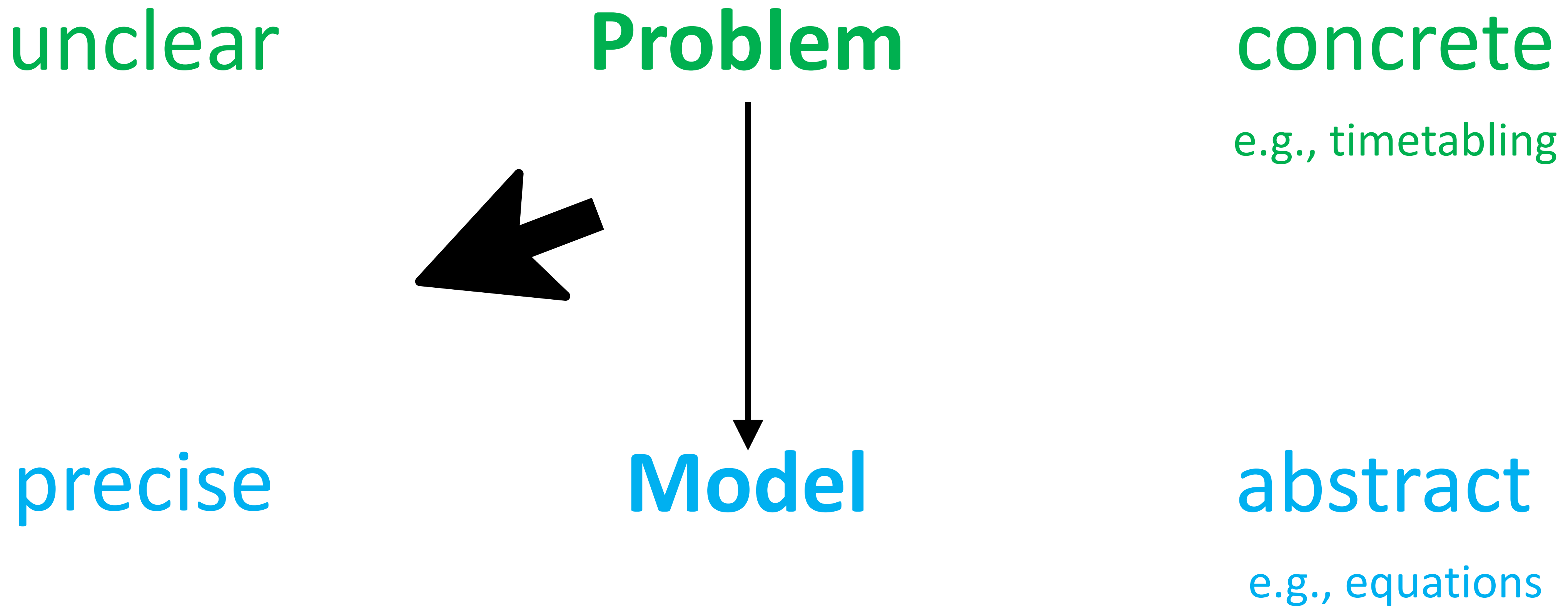
concrete

e.g., timetabling

Model

abstract

e.g., equations


unclear
Problem
concrete
e.g., timetabling
precise
Model
abstract
e.g., equations

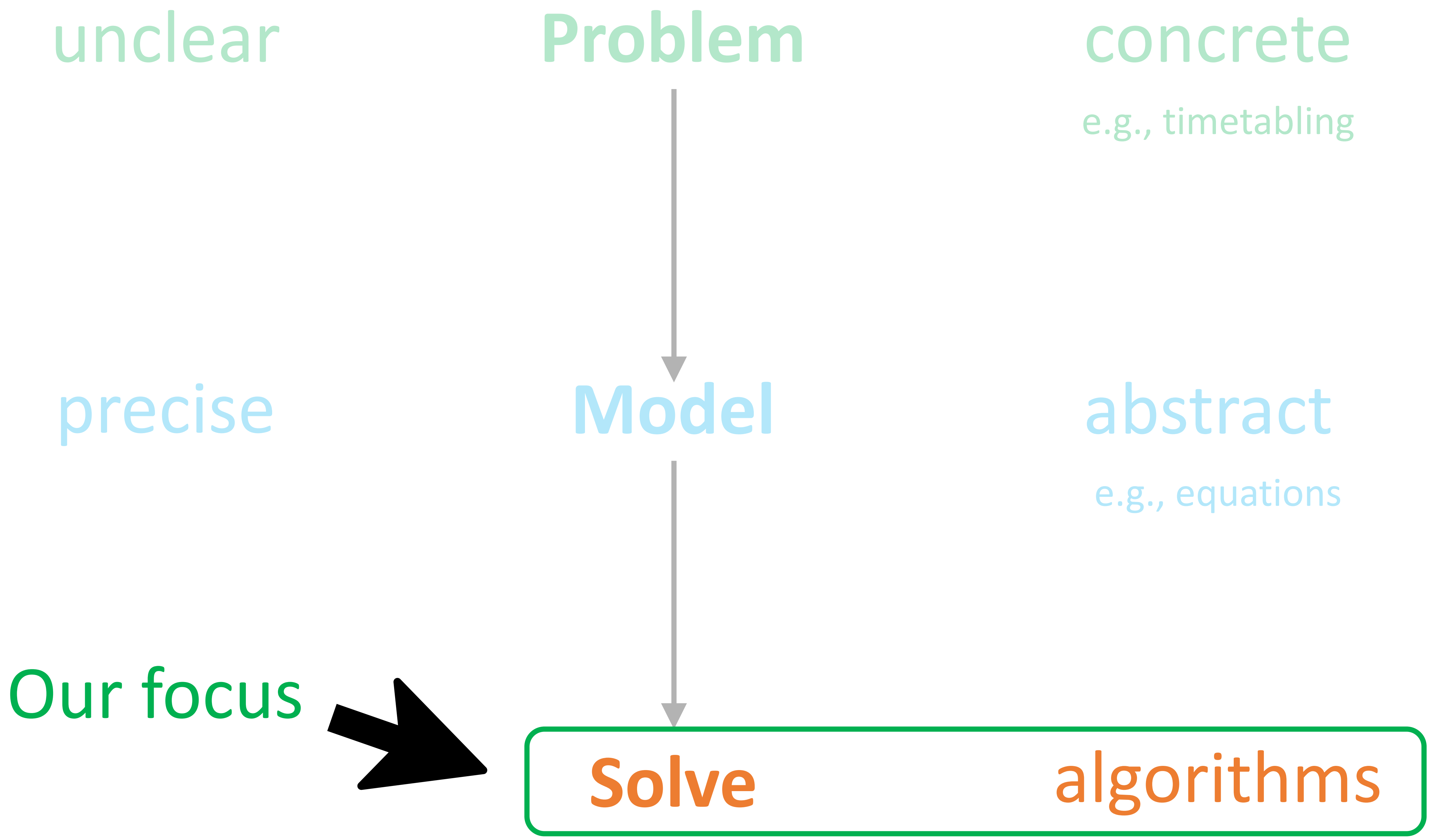
unclear **Problem** concrete

e.g., timetabling

precise **Model** abstract

e.g., equations

Solve algorithms

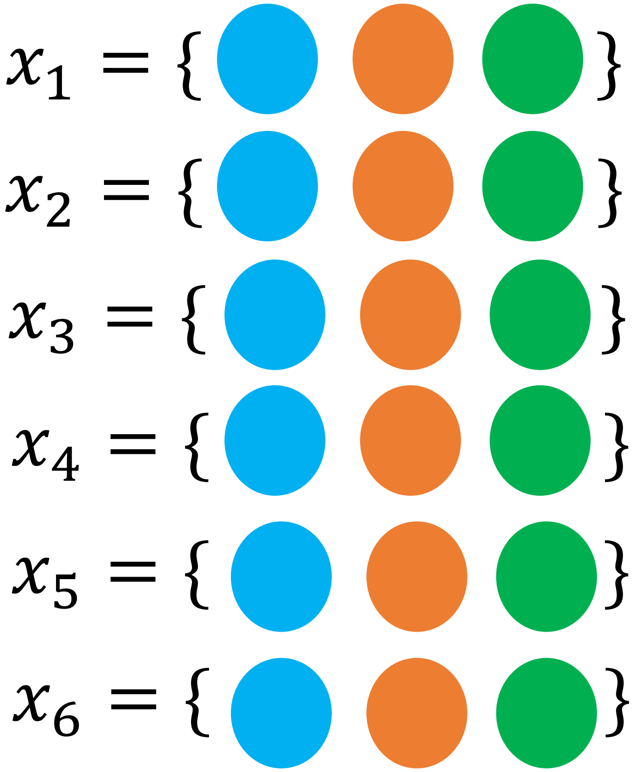

unclear
Problem
concrete
e.g., timetabling
precise
Model
abstract
e.g., equations
Our focus
Solve
algorithms

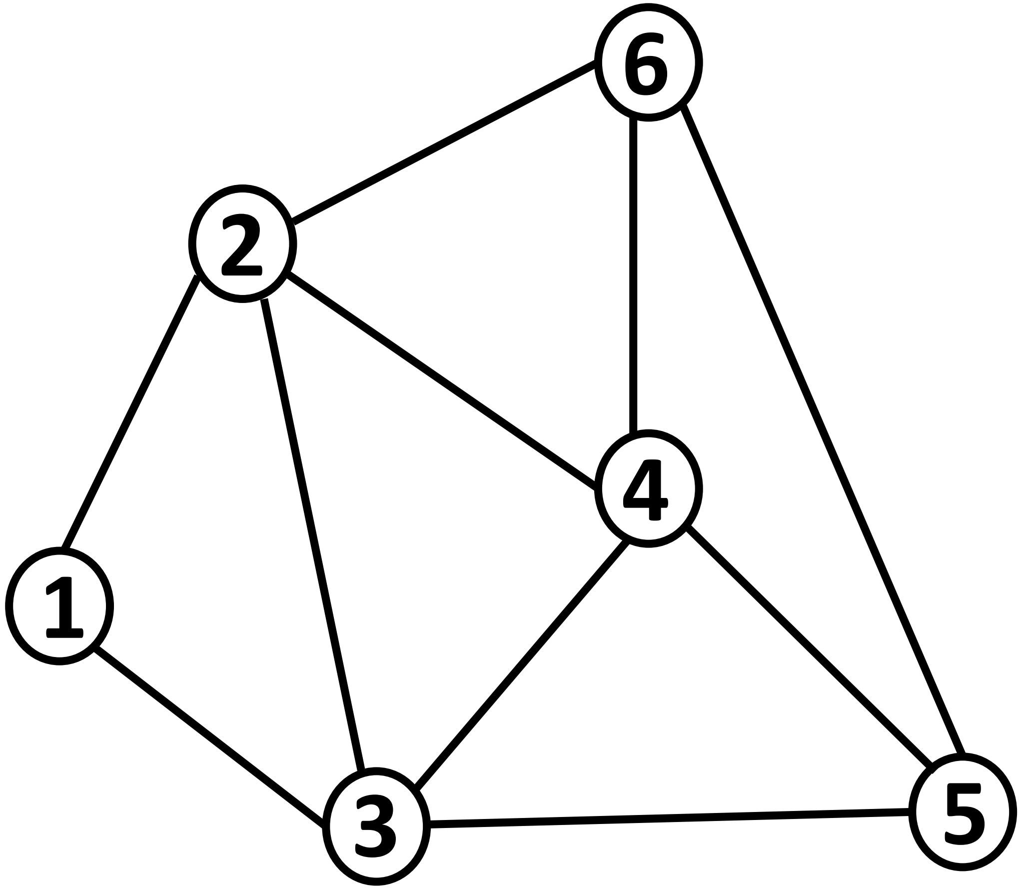
Search

Branching + Propagation

Graph Colouring

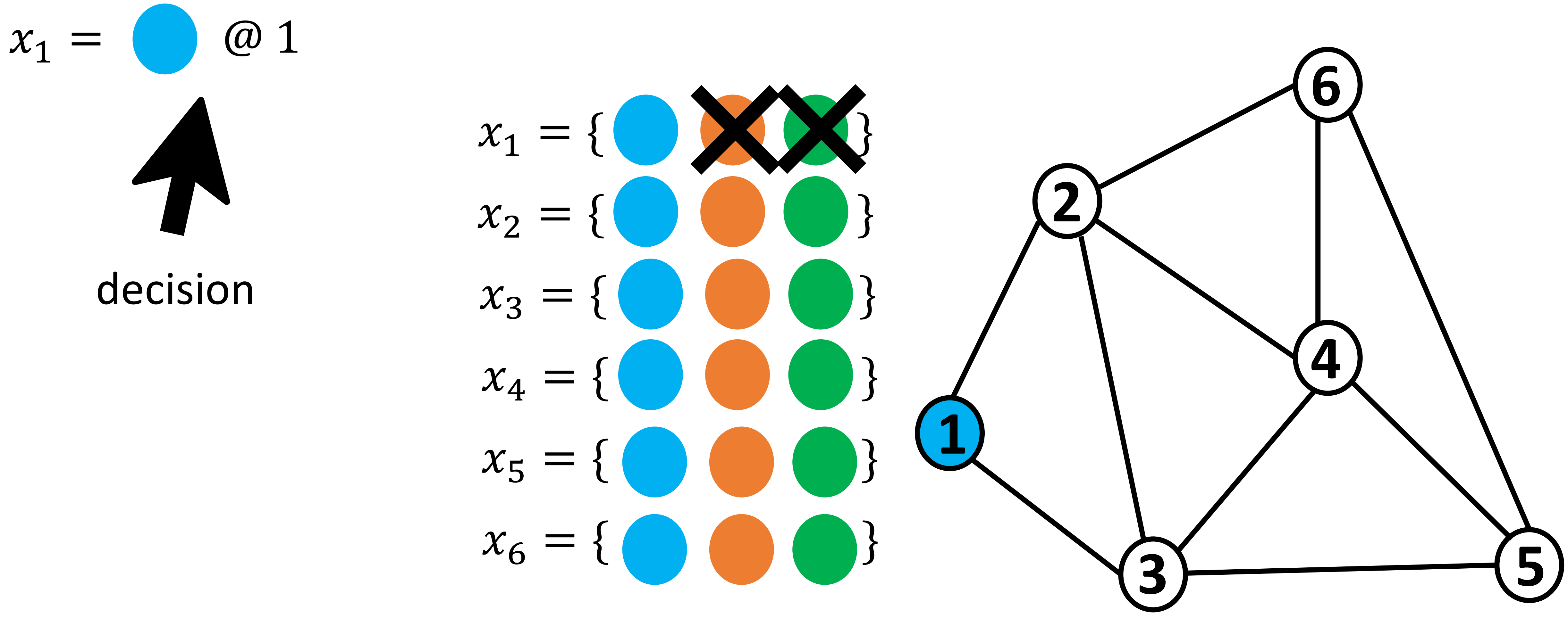
$x_i \neq x_j$





Graph Colouring

$x_i \neq x_j$



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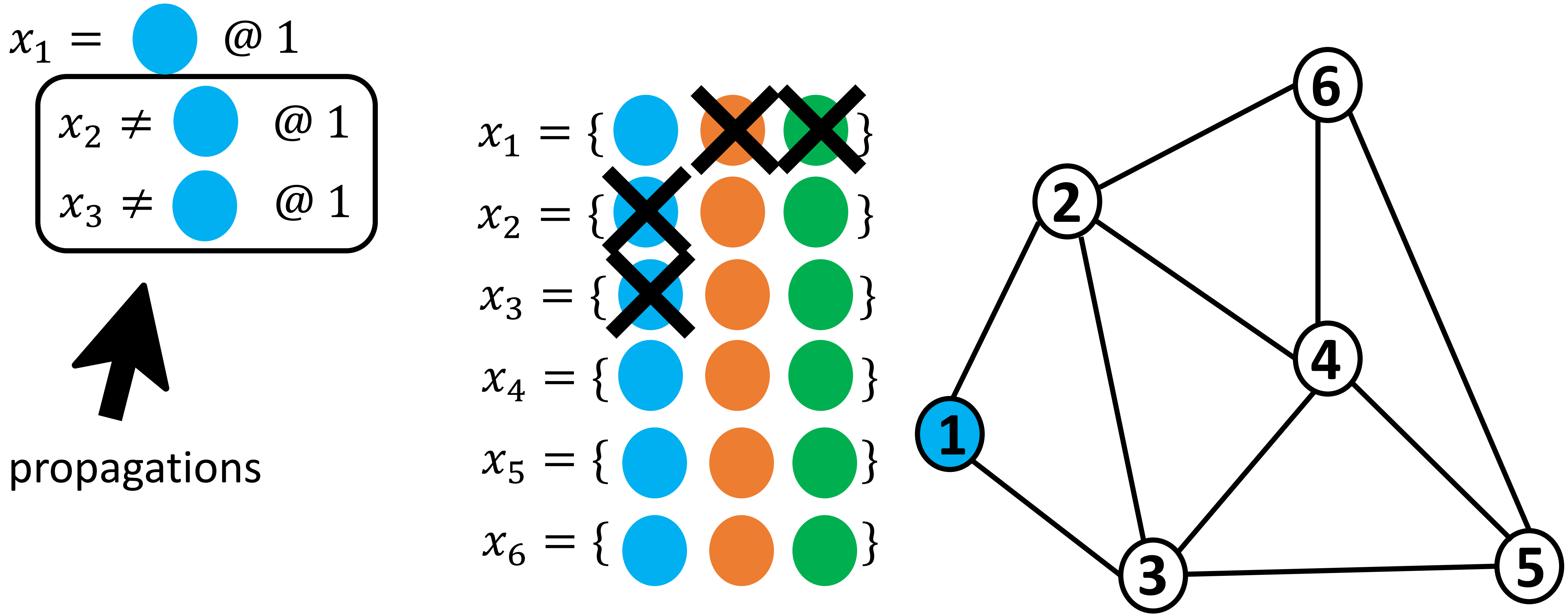

Graph Colouring

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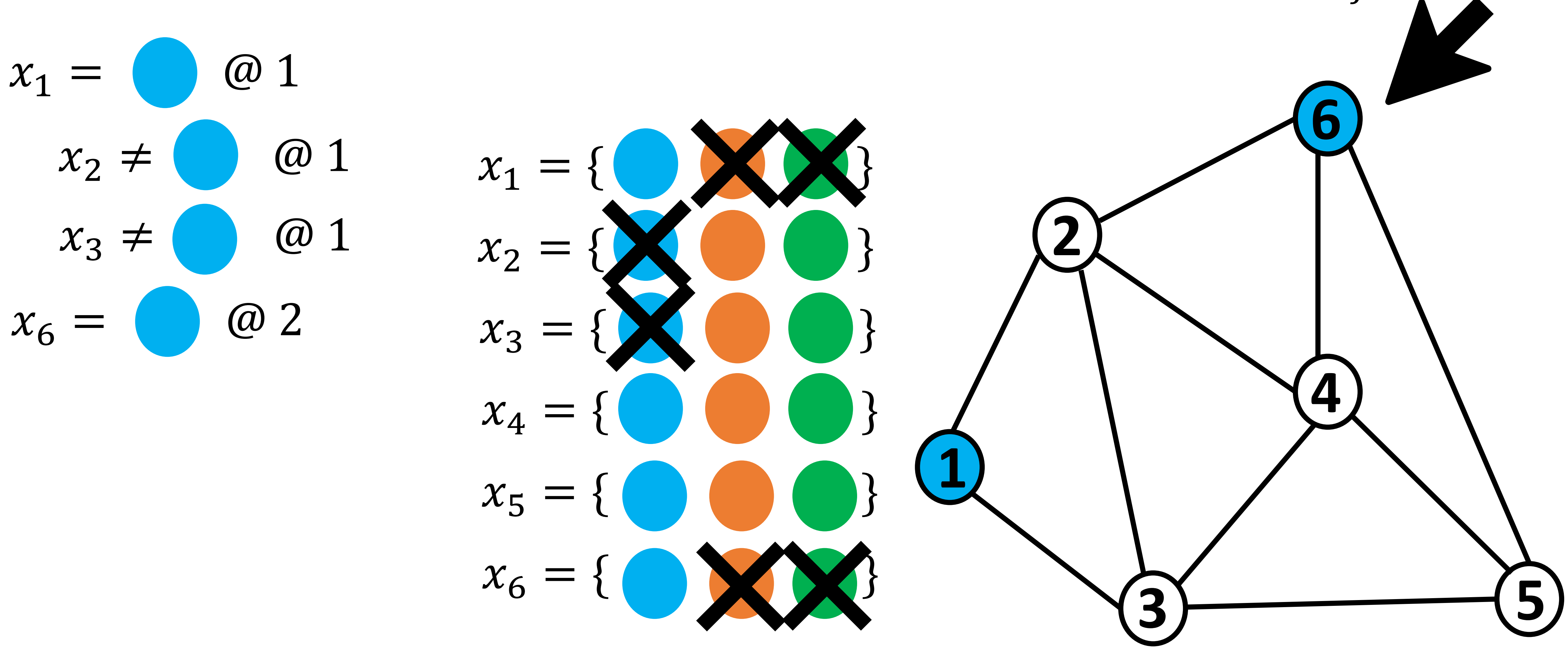

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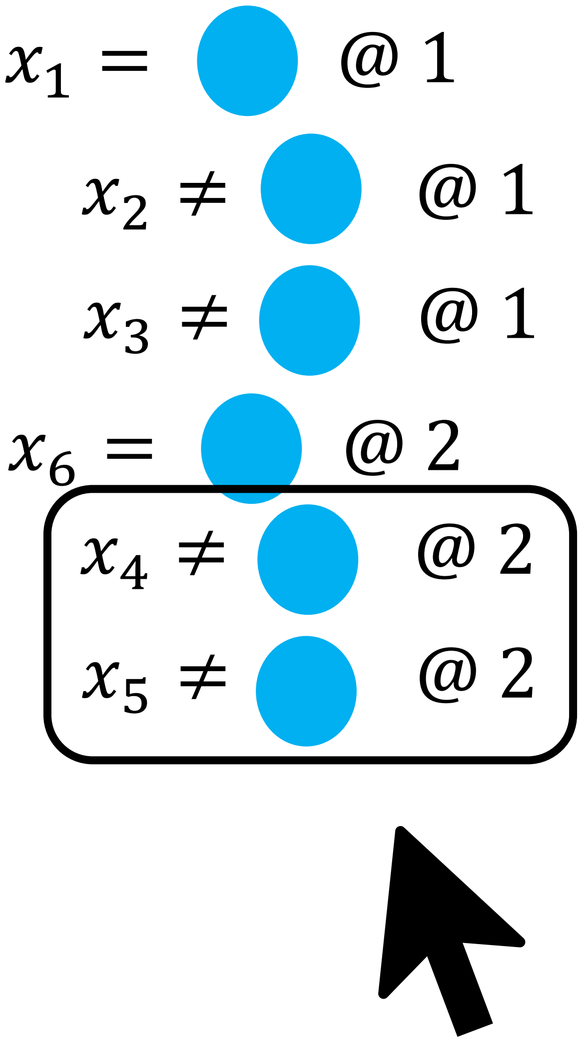

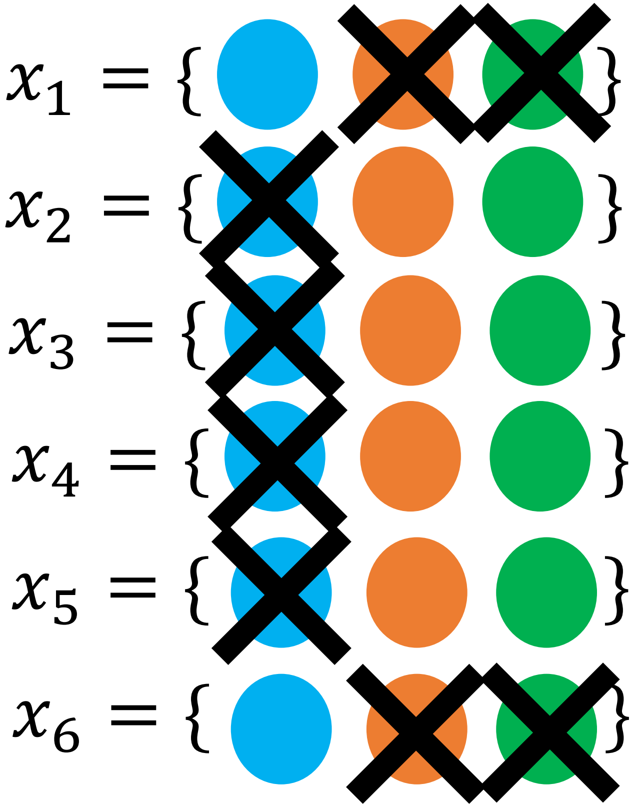

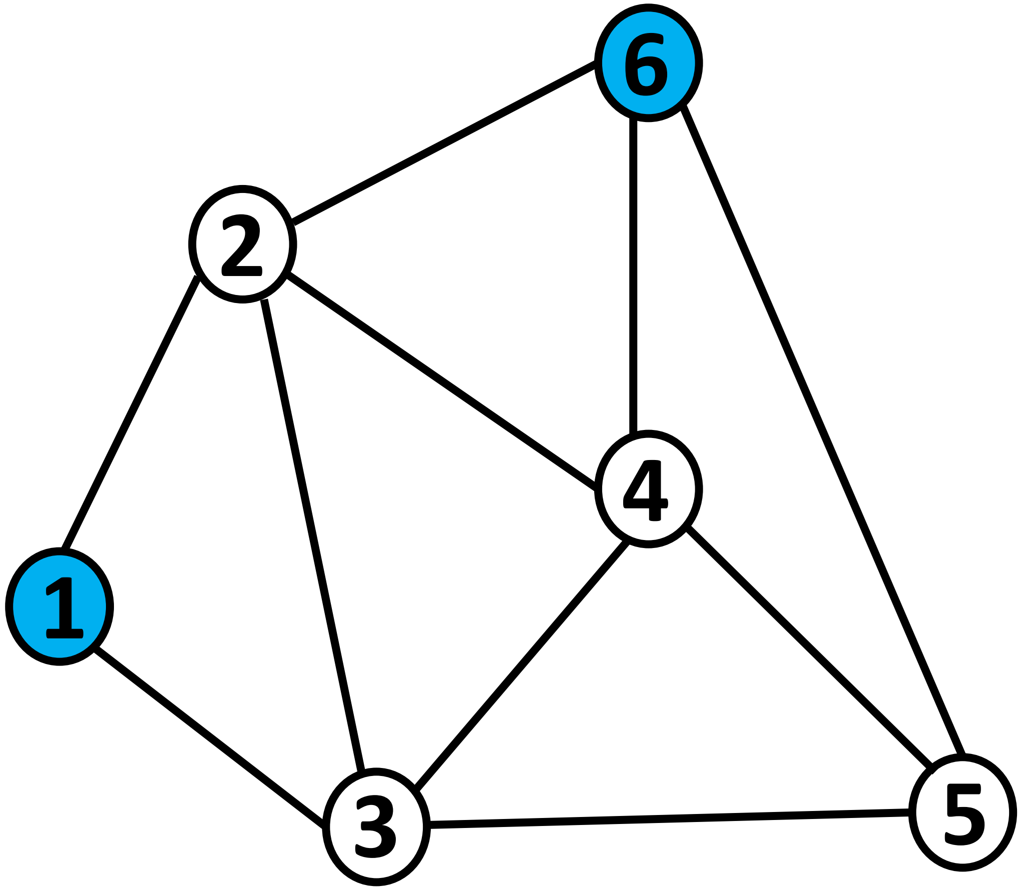

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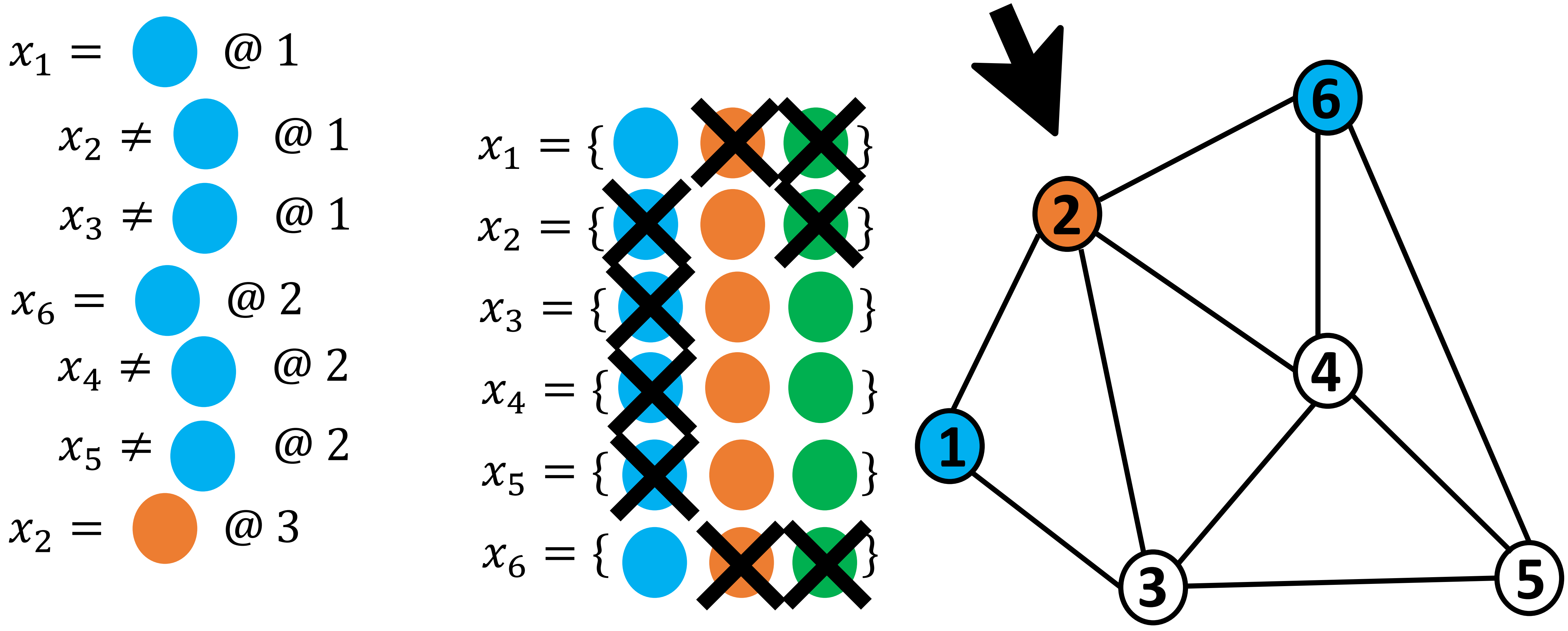

Graph Colouring

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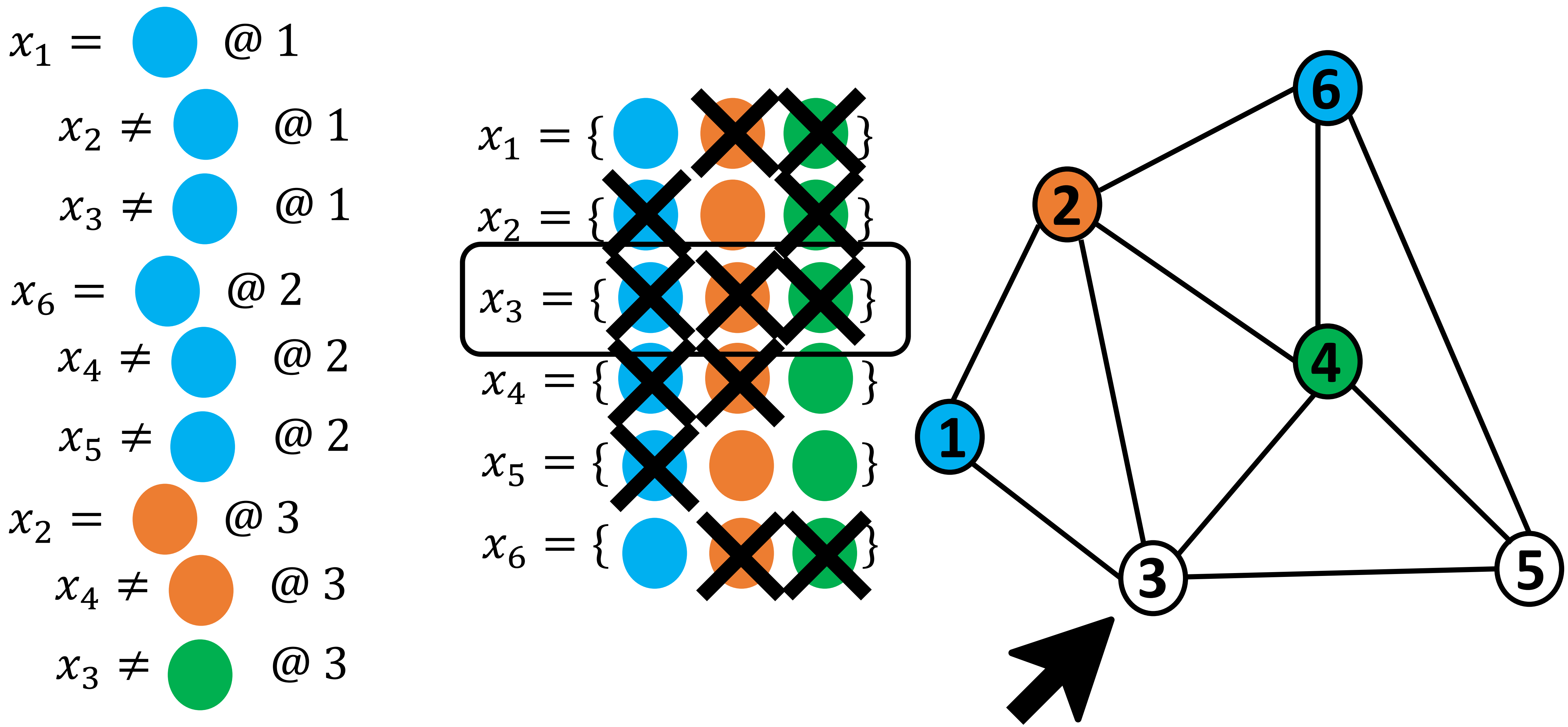
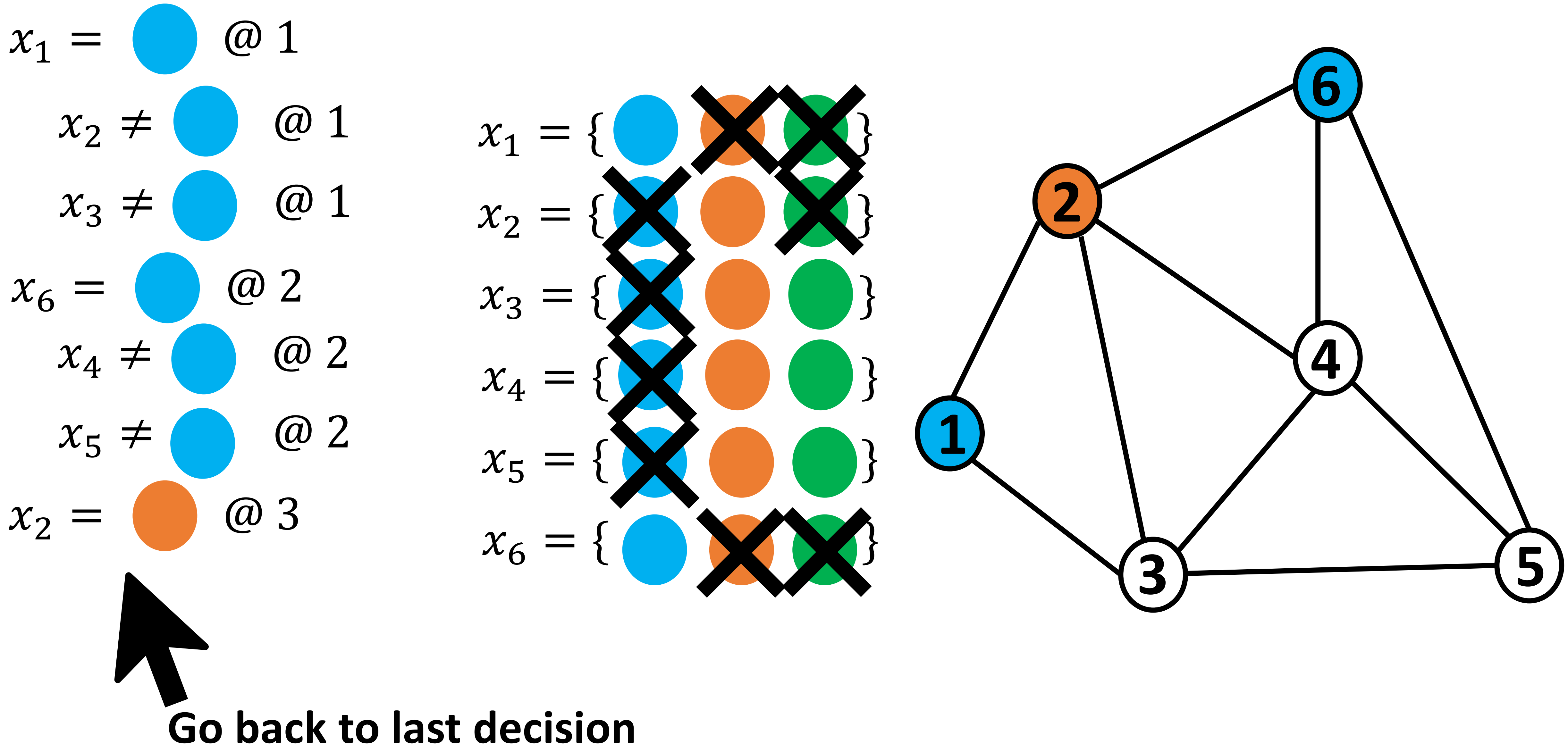


No colours left, conflict!

Graph Colouring

$x_i \neq x_j$



Graph Colouring

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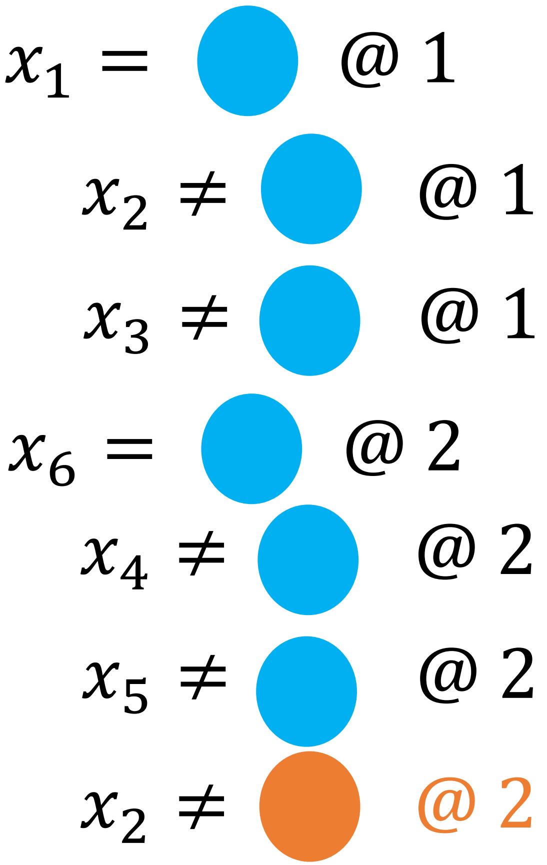
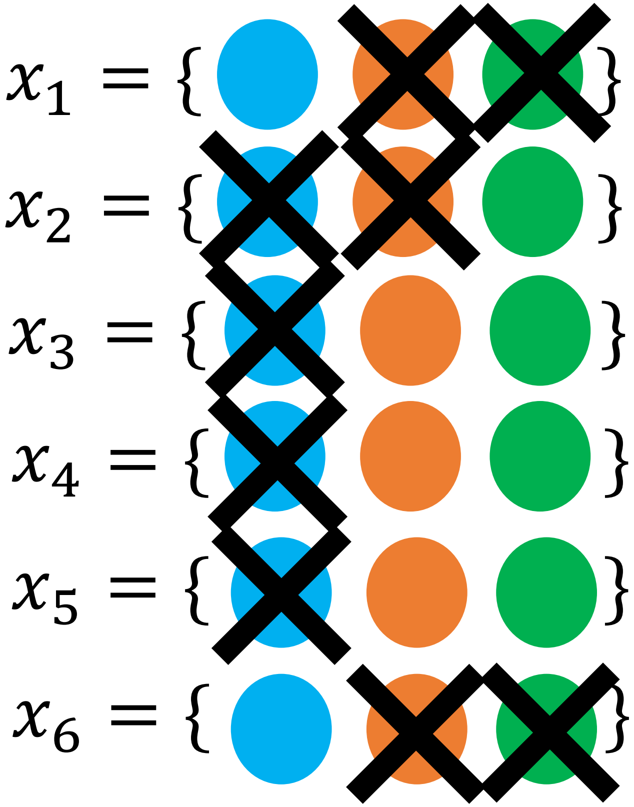
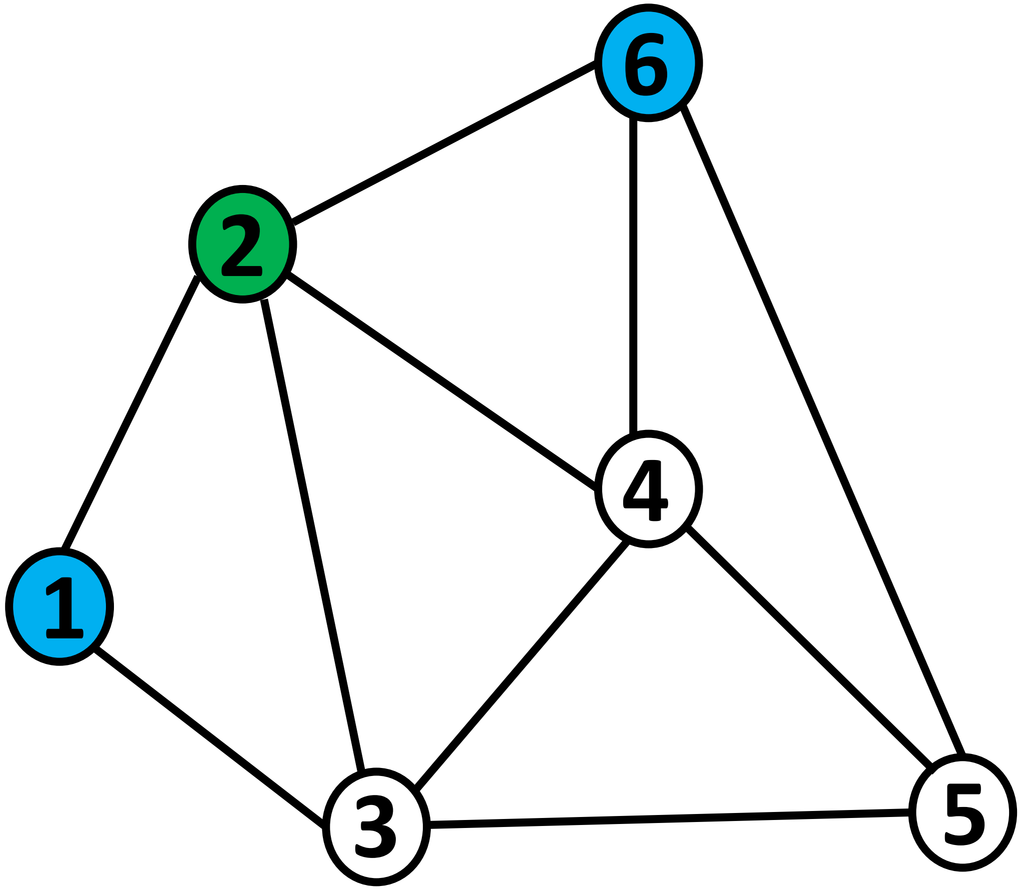


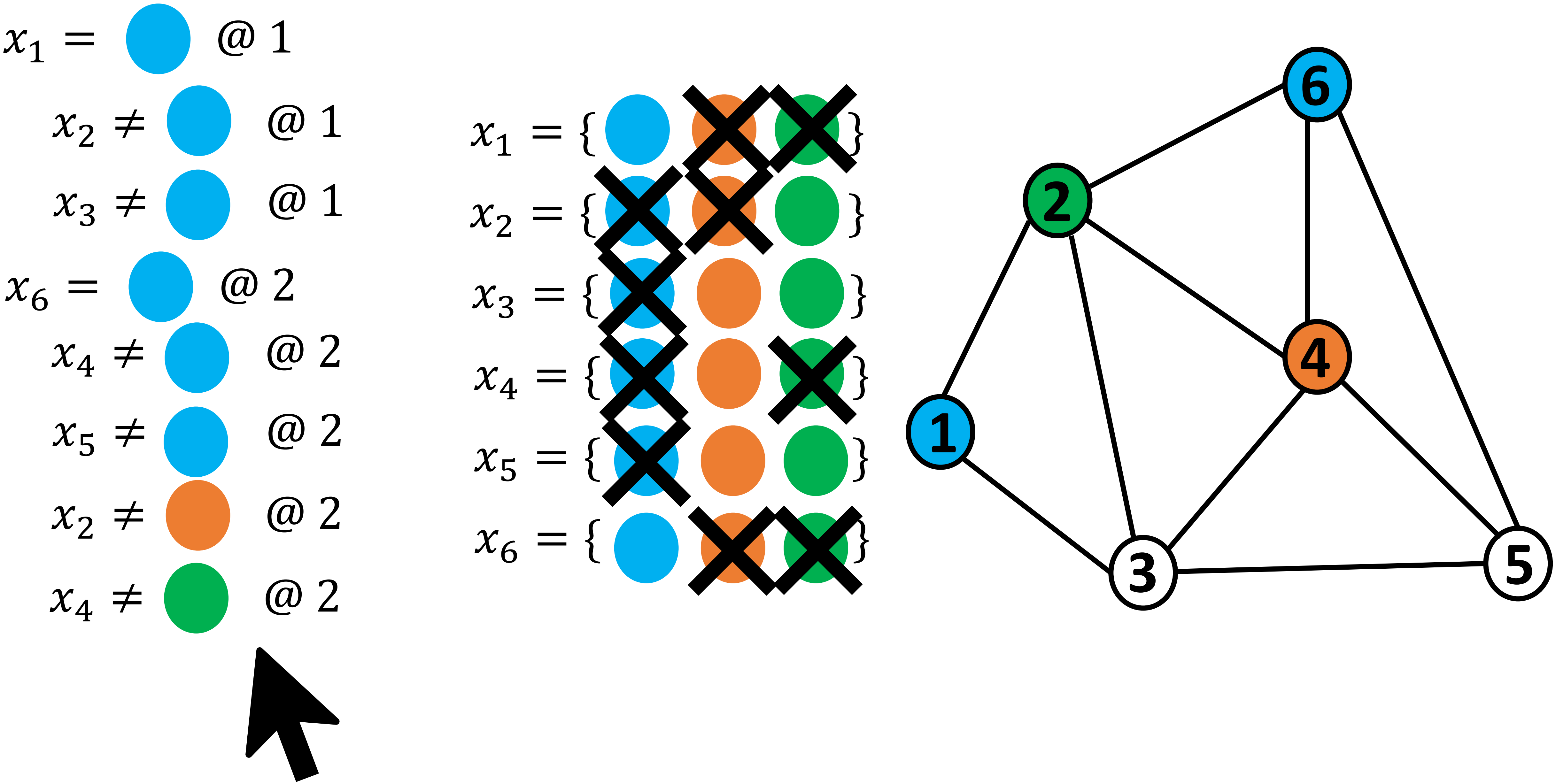




We now know that orange was in fact not possible

Graph Colouring

$x_i \neq x_j$



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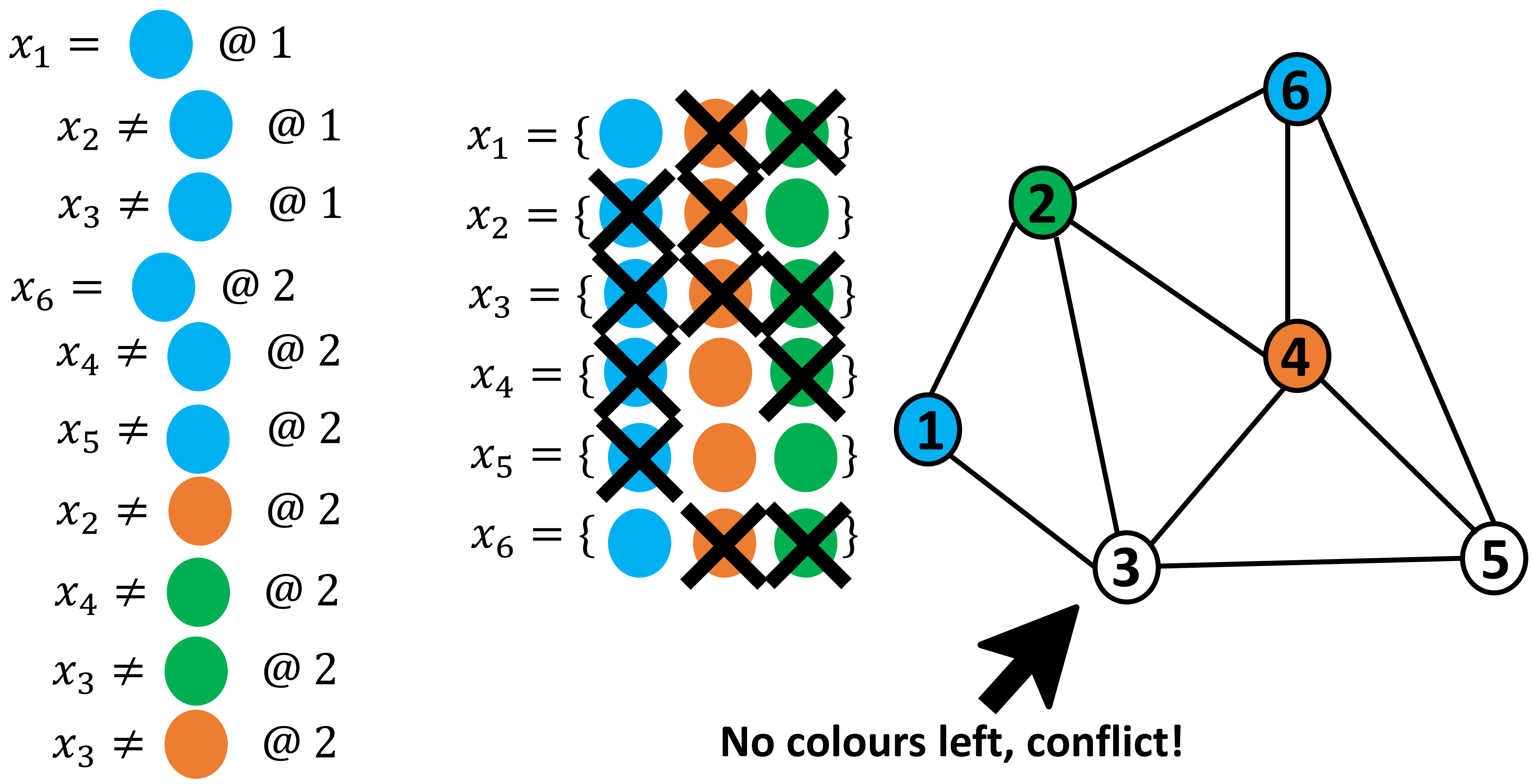

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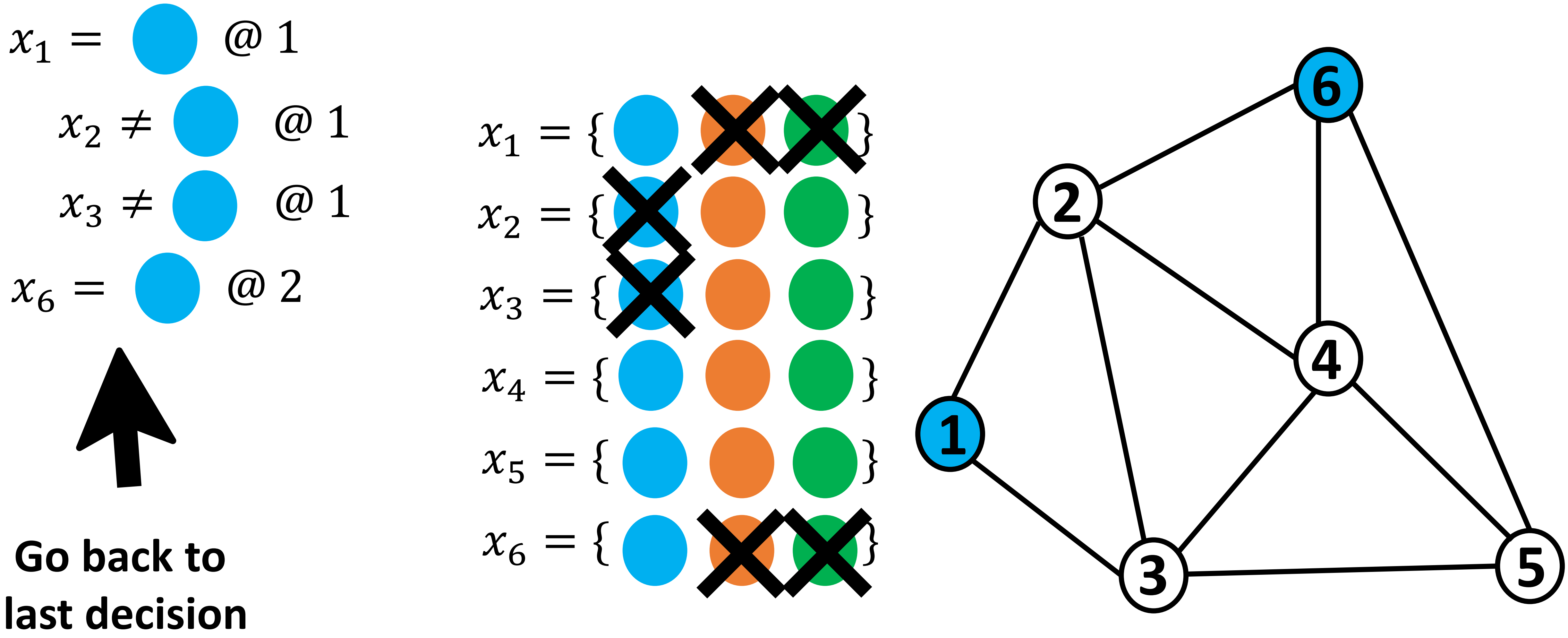

Graph Colouring

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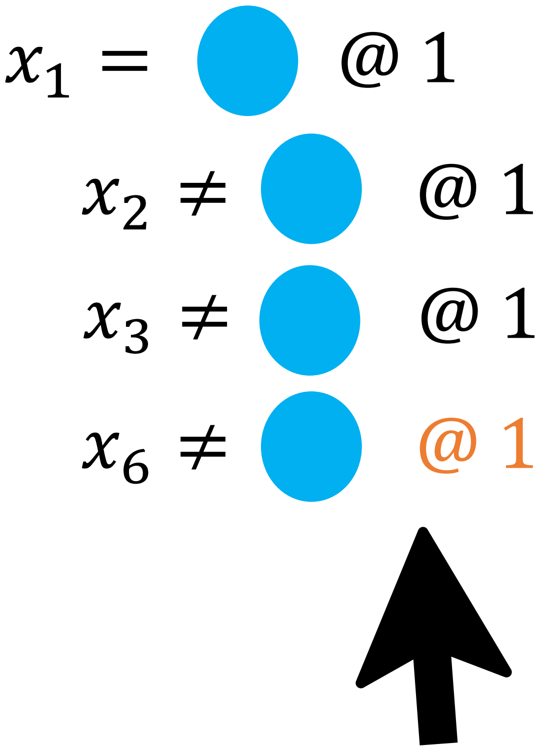
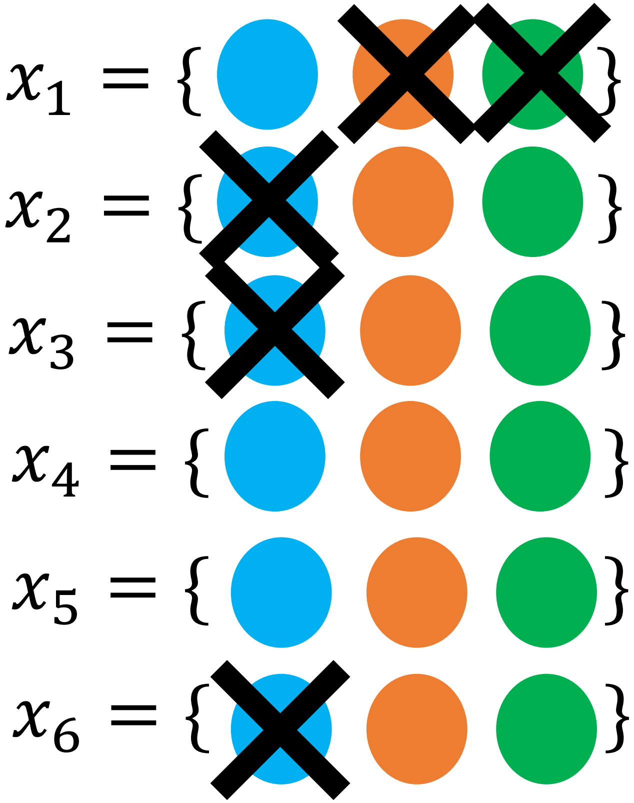
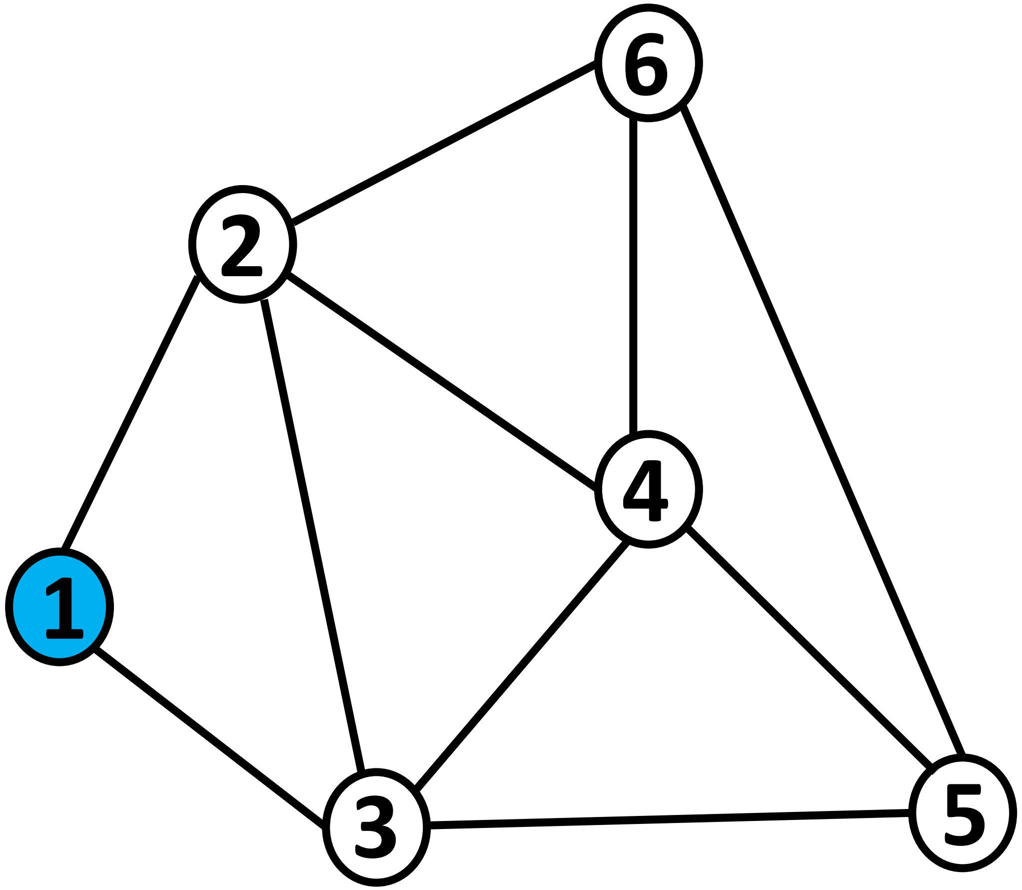


We now know that blue was in fact not possible





Graph Colouring

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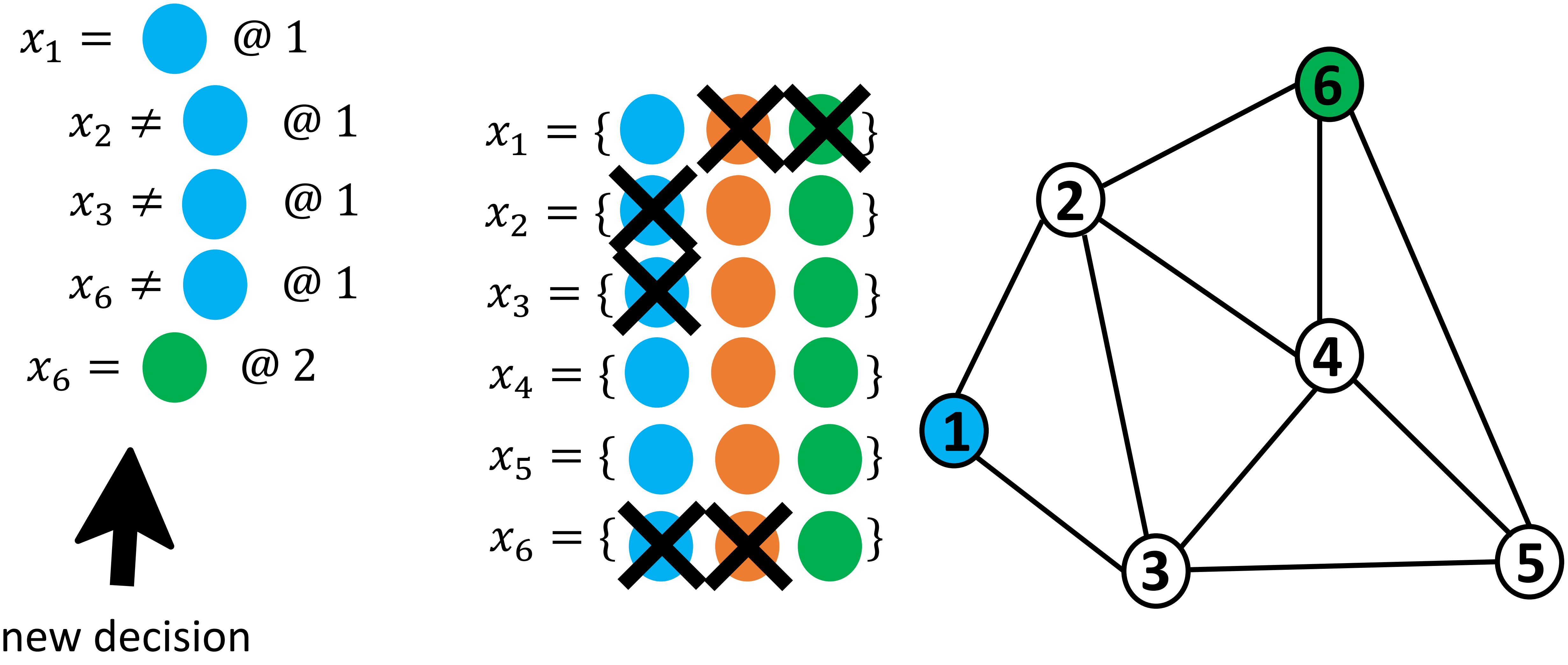

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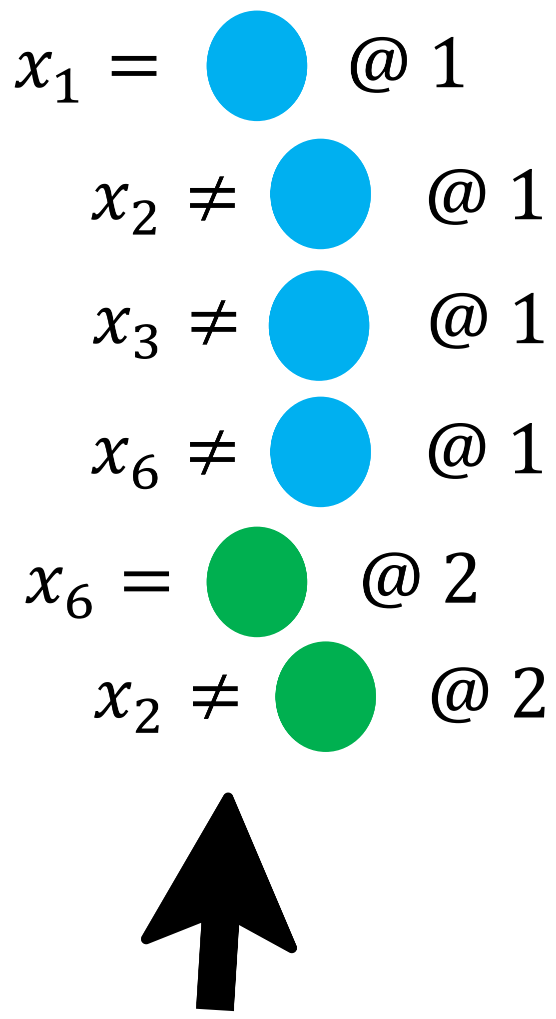
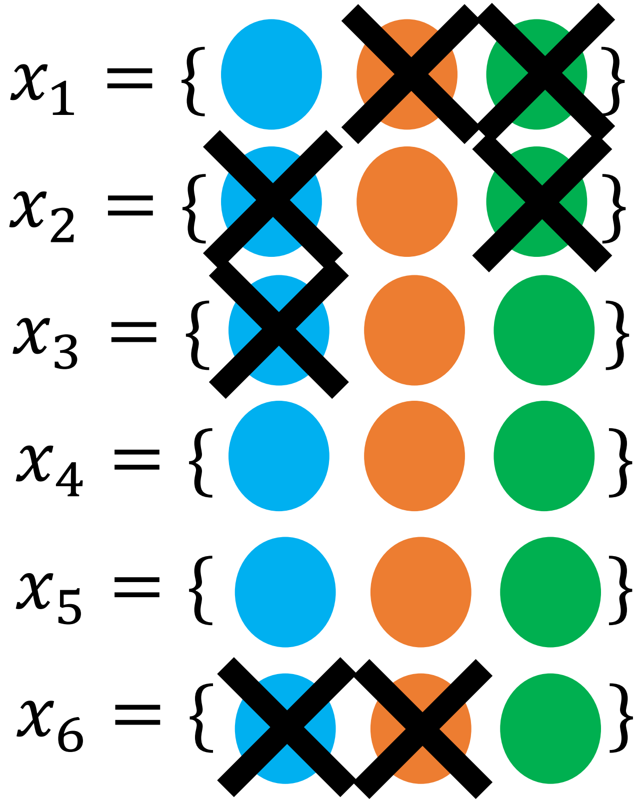
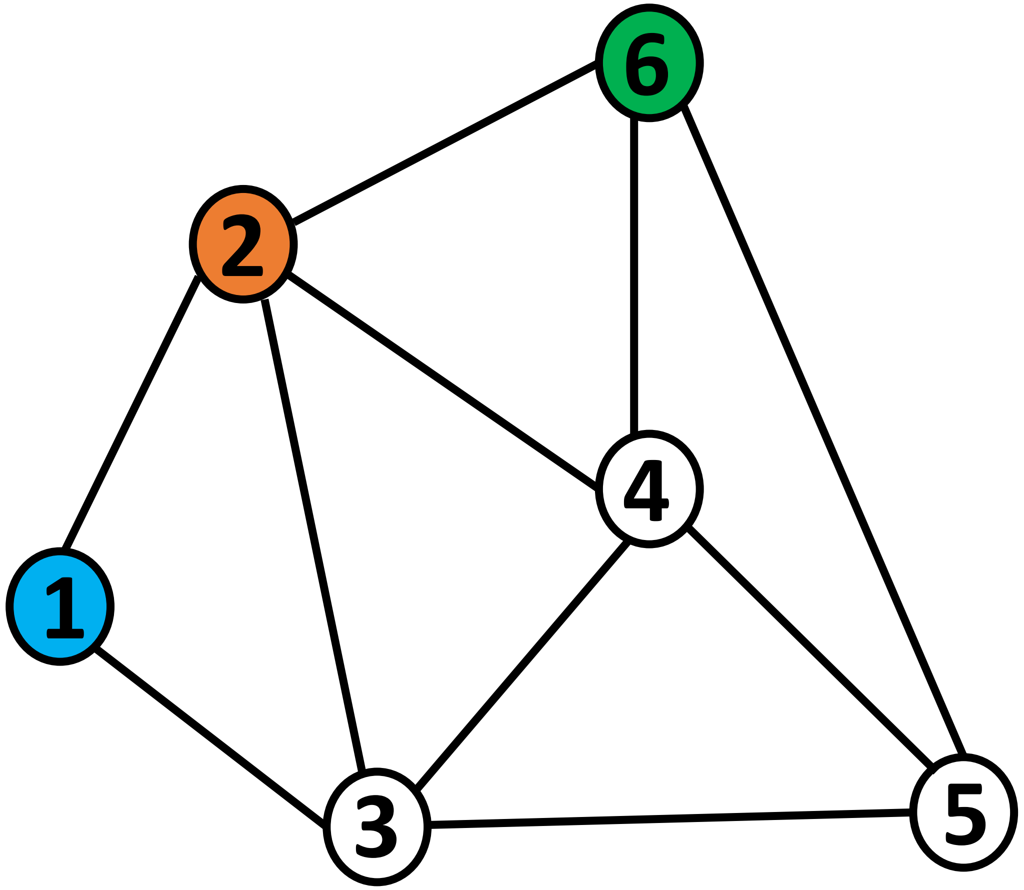
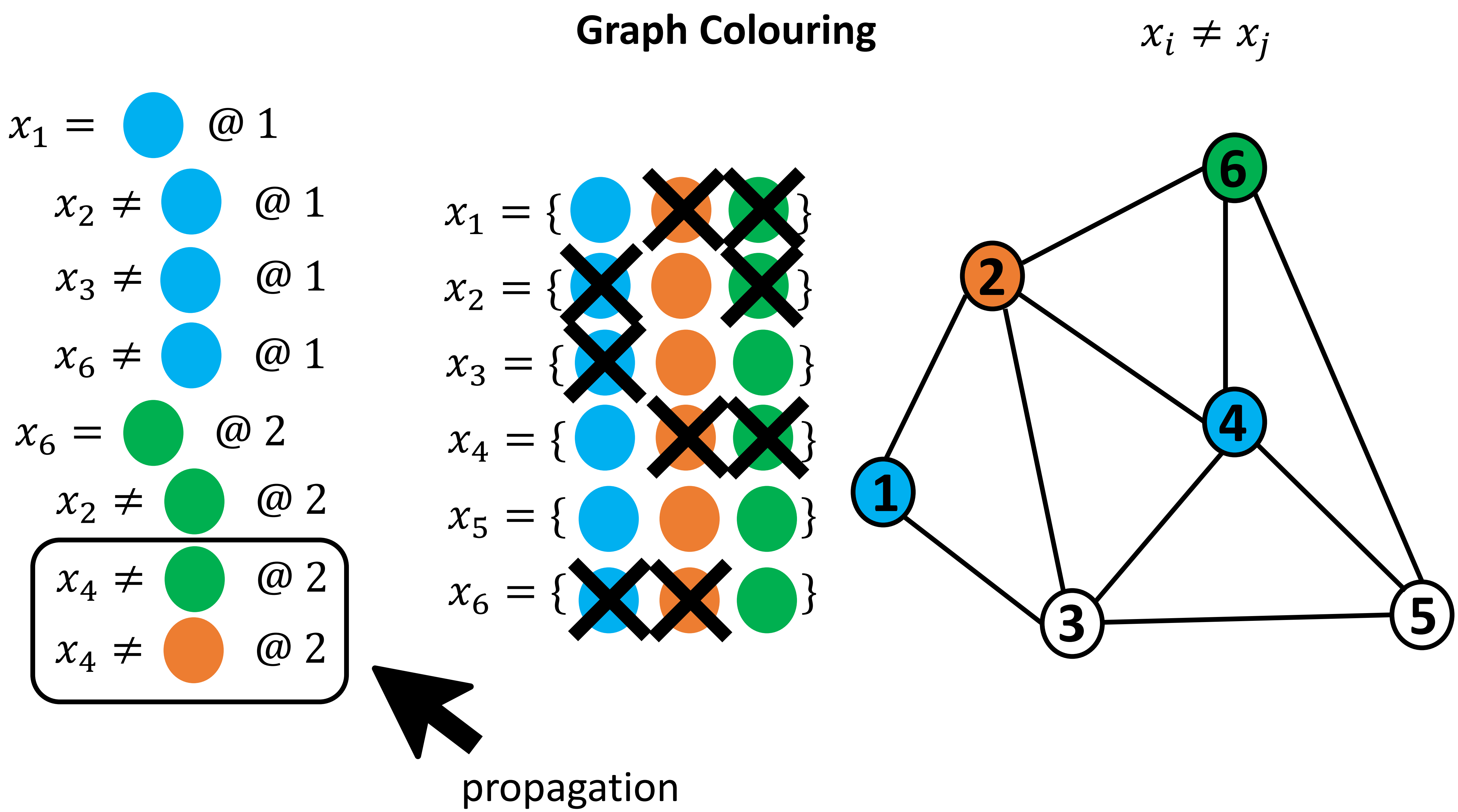


propagation

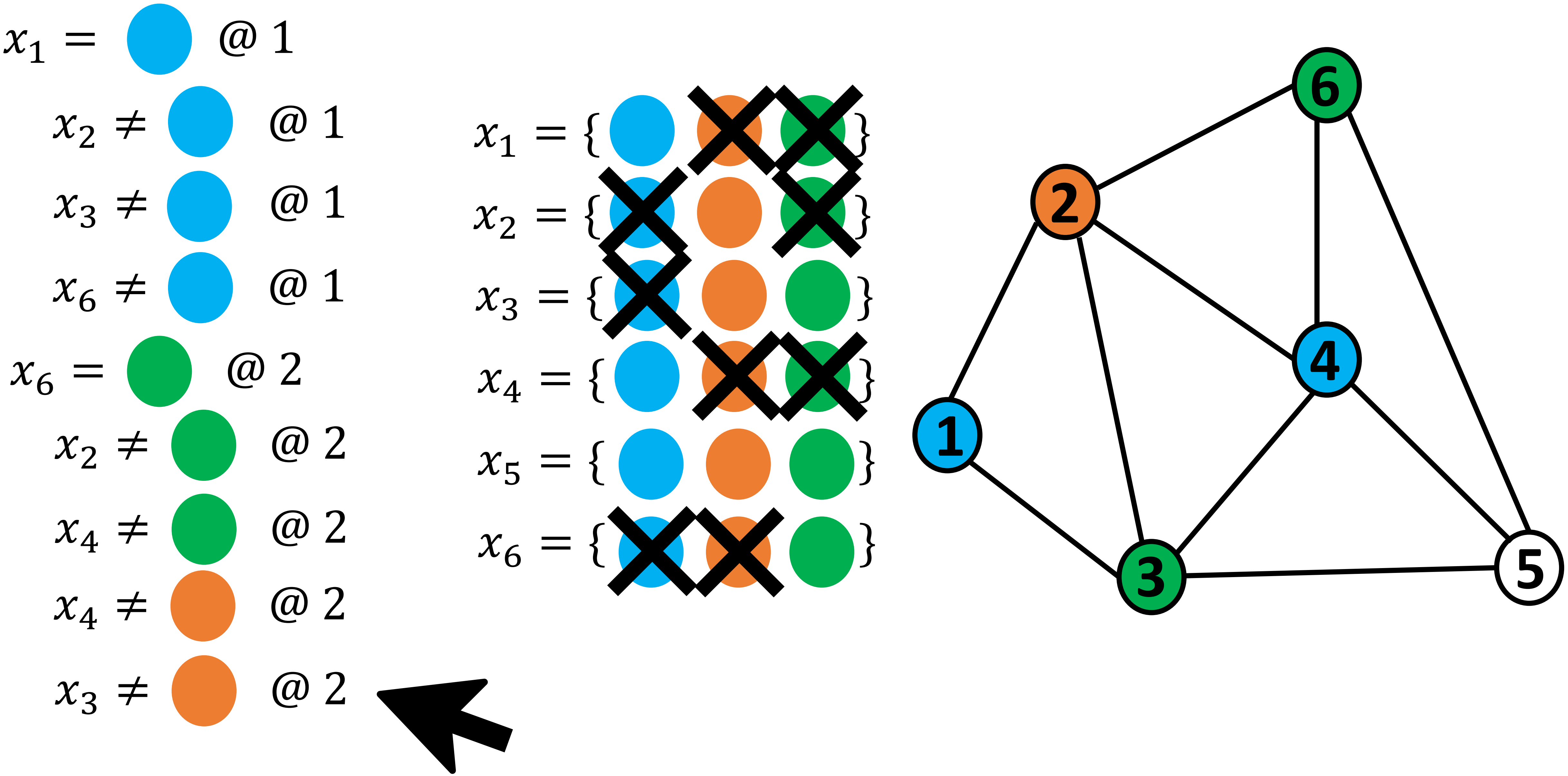





Graph Colouring
$x_i \neq x_j$
$x_1 =$ @ 1
$x_2 \neq$ @ 1
$x_3 \neq$ @ 1
$x_6 \neq$ @ 1
$x_6 =$ @ 2
$x_2 \neq$ @ 2
$x_4 \neq$ @ 2
$x_4 \neq$ @ 2
propagation
$x_1 = \{ \}$
$x_2 = \{ \}$
$x_3 = \{ \}$
$x_4 = \{ \}$
$x_5 = \{ \}$
$x_6 = \{ \}$
1
2
3
4
5
6

Graph Colouring

$x_i \neq x_j$



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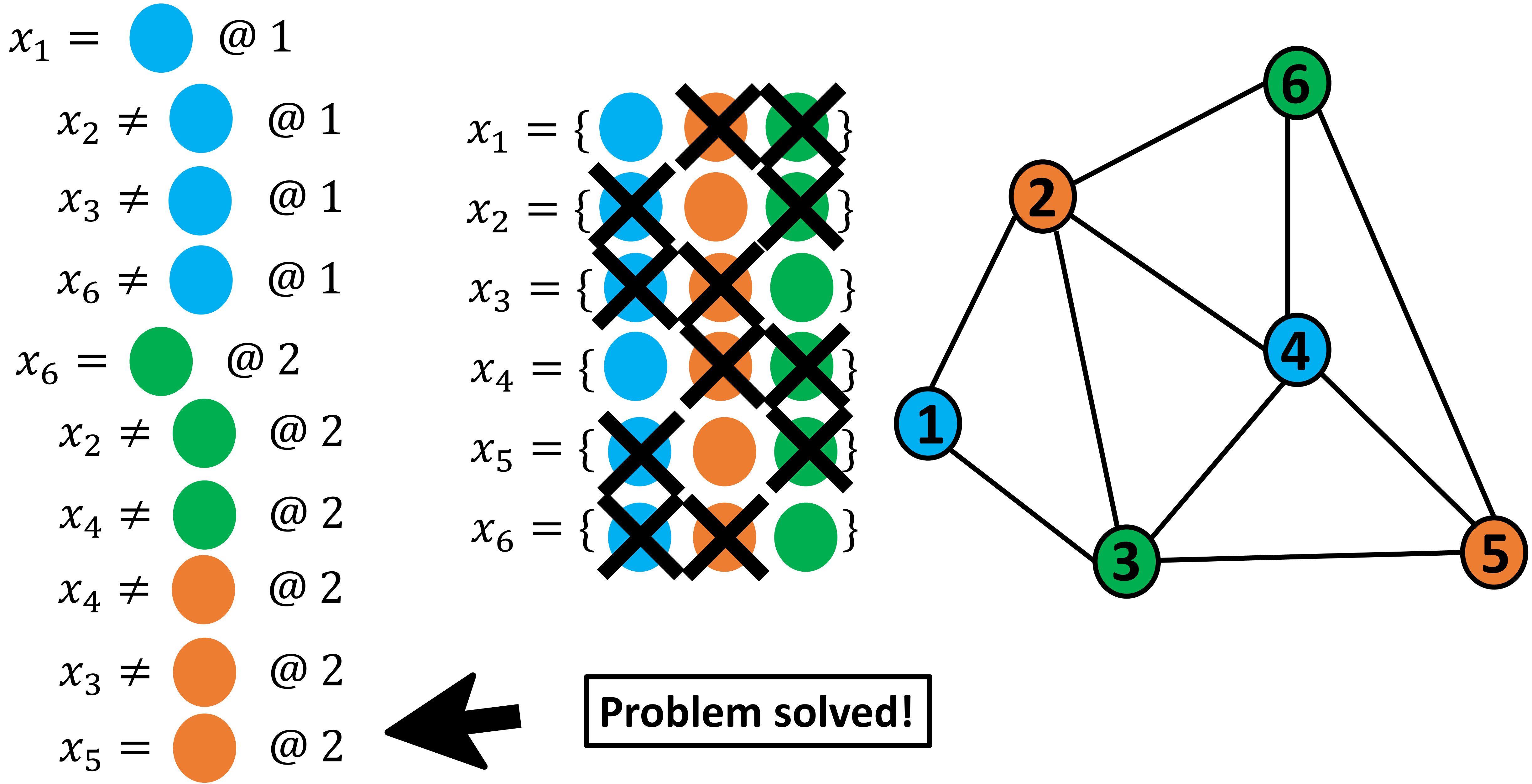
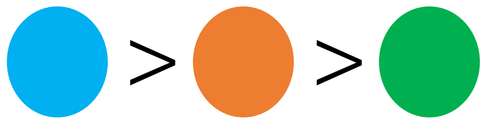
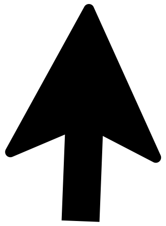


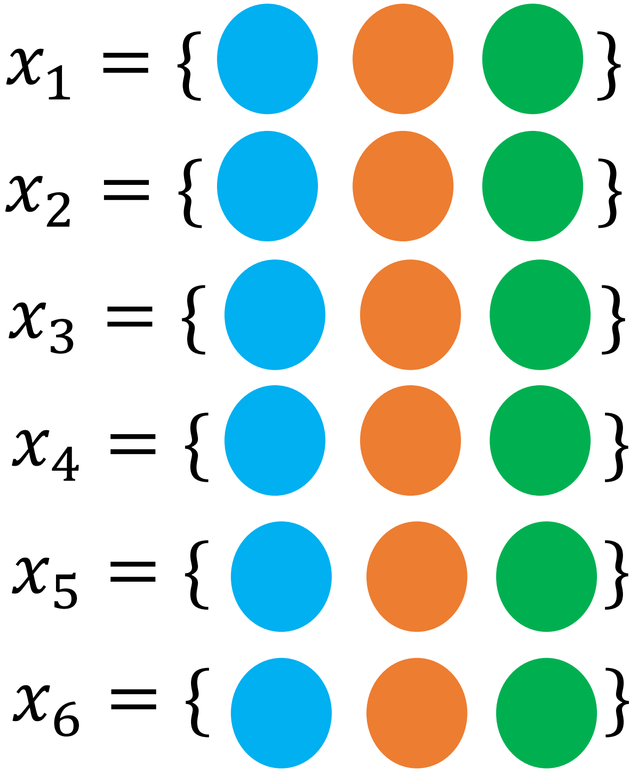
Graph Colouring

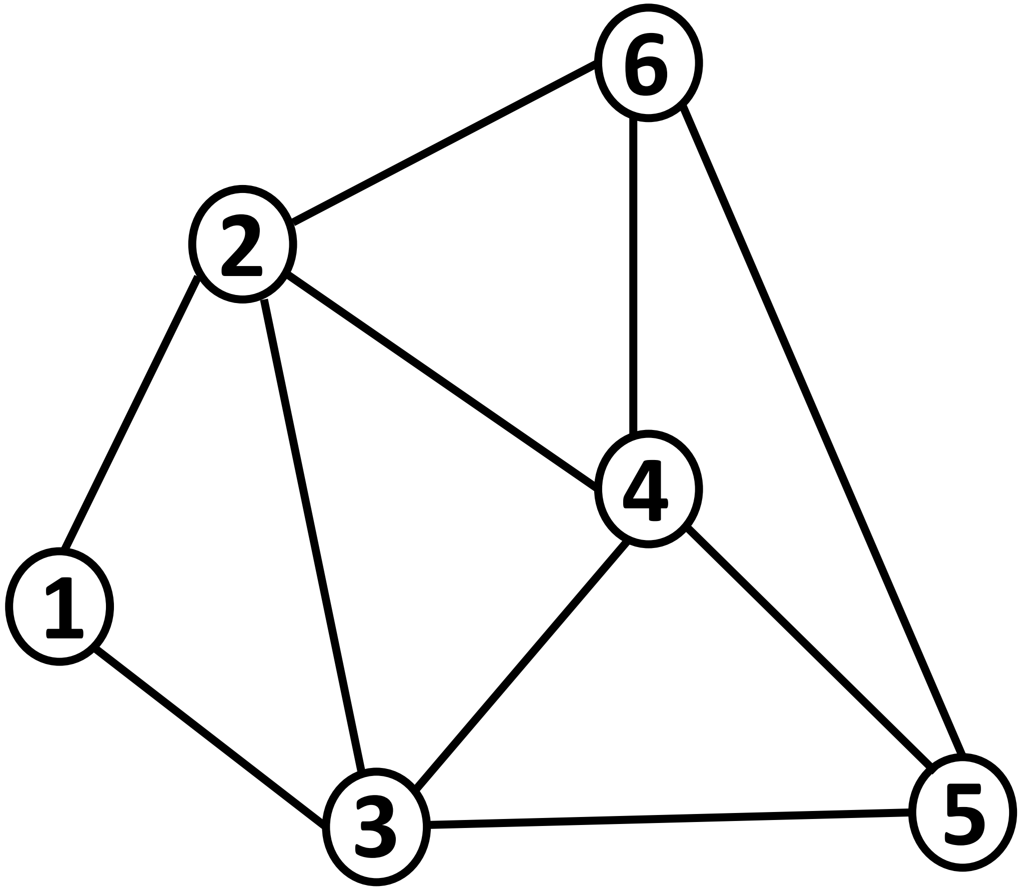


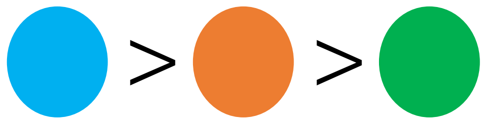
$x_1 > x_2 > \cdots > x_6$



$x_i \neq x_j$



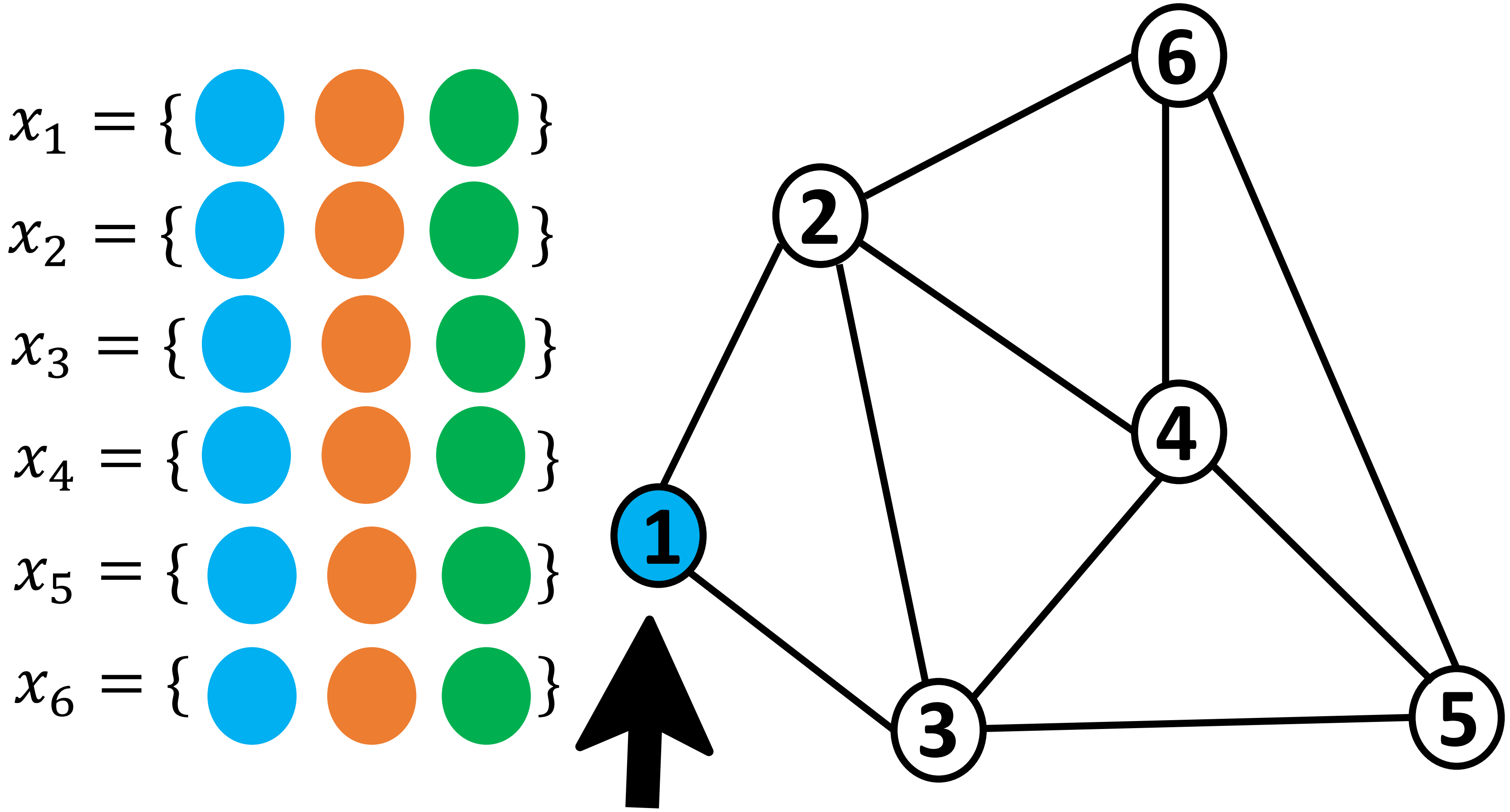


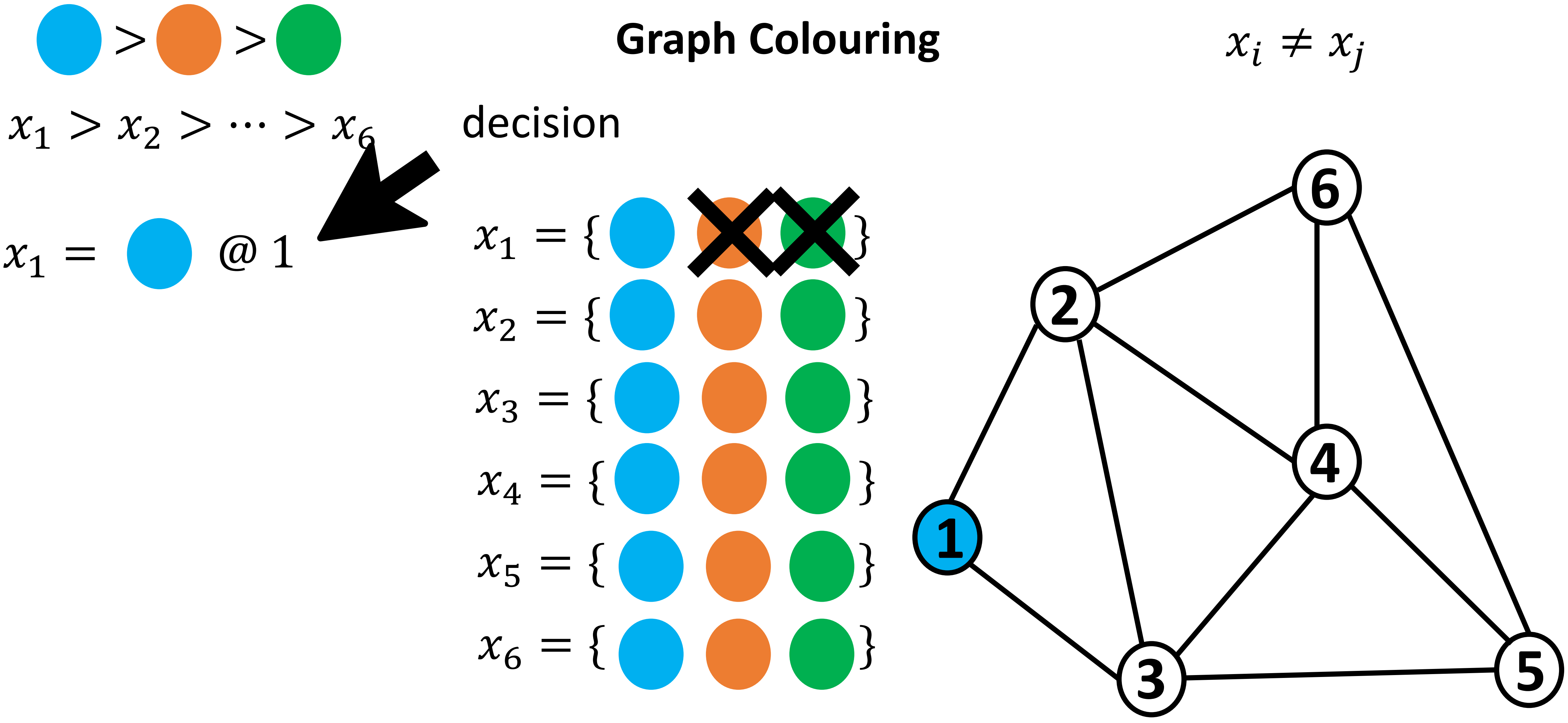


$x_1 > x_2 > \cdots > x_6$

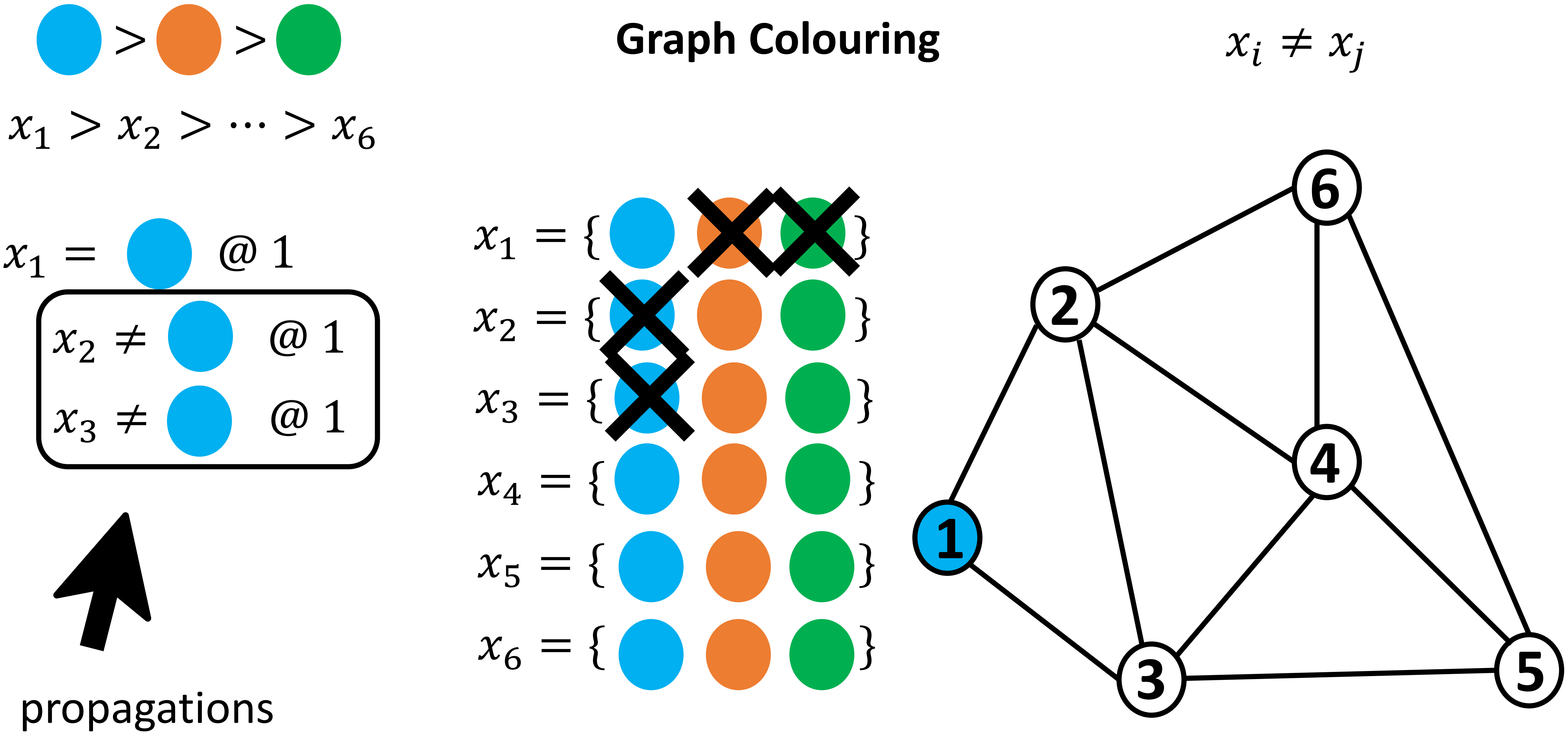
Graph Colouring

$x_i \neq x_j$

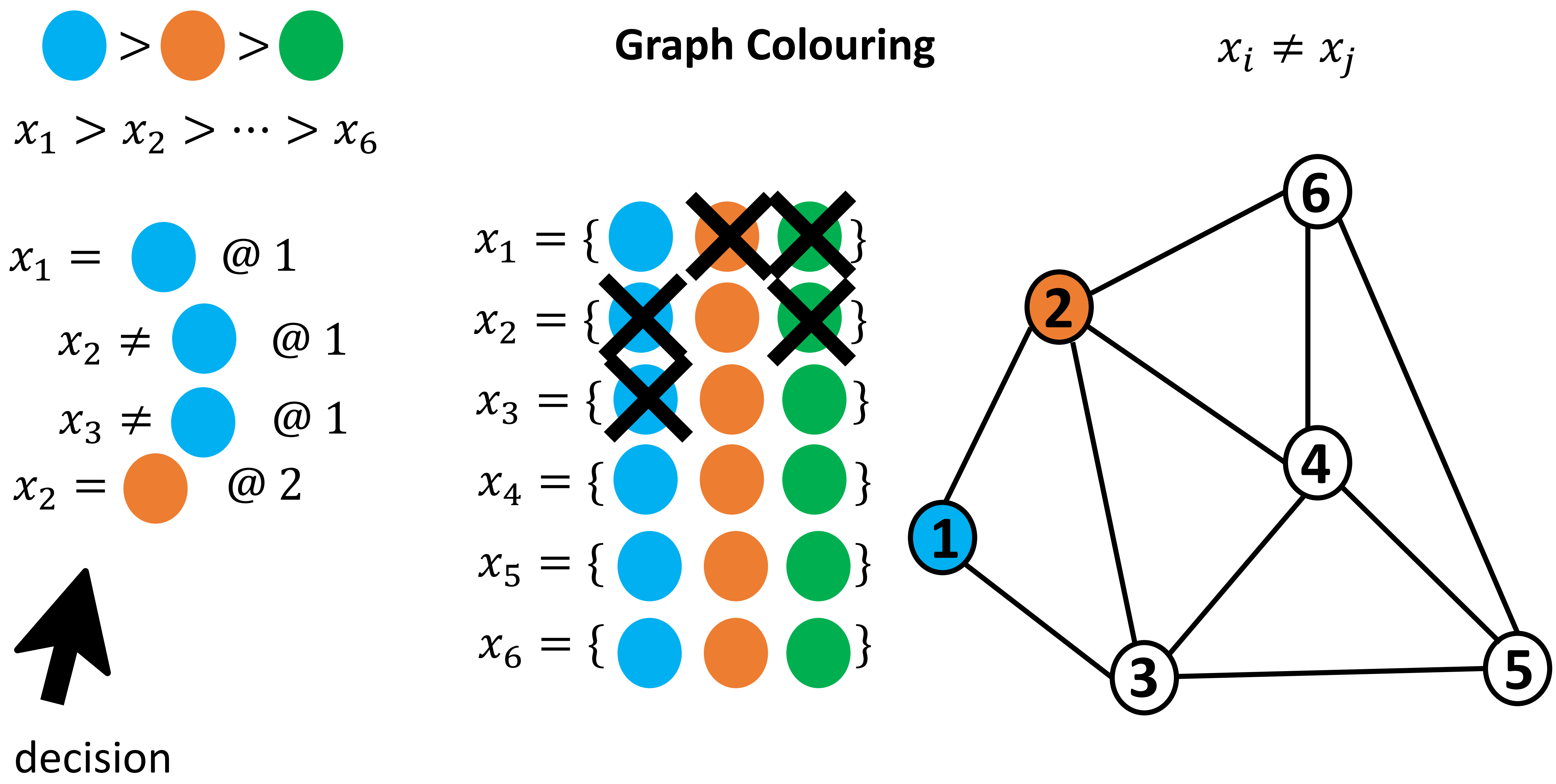


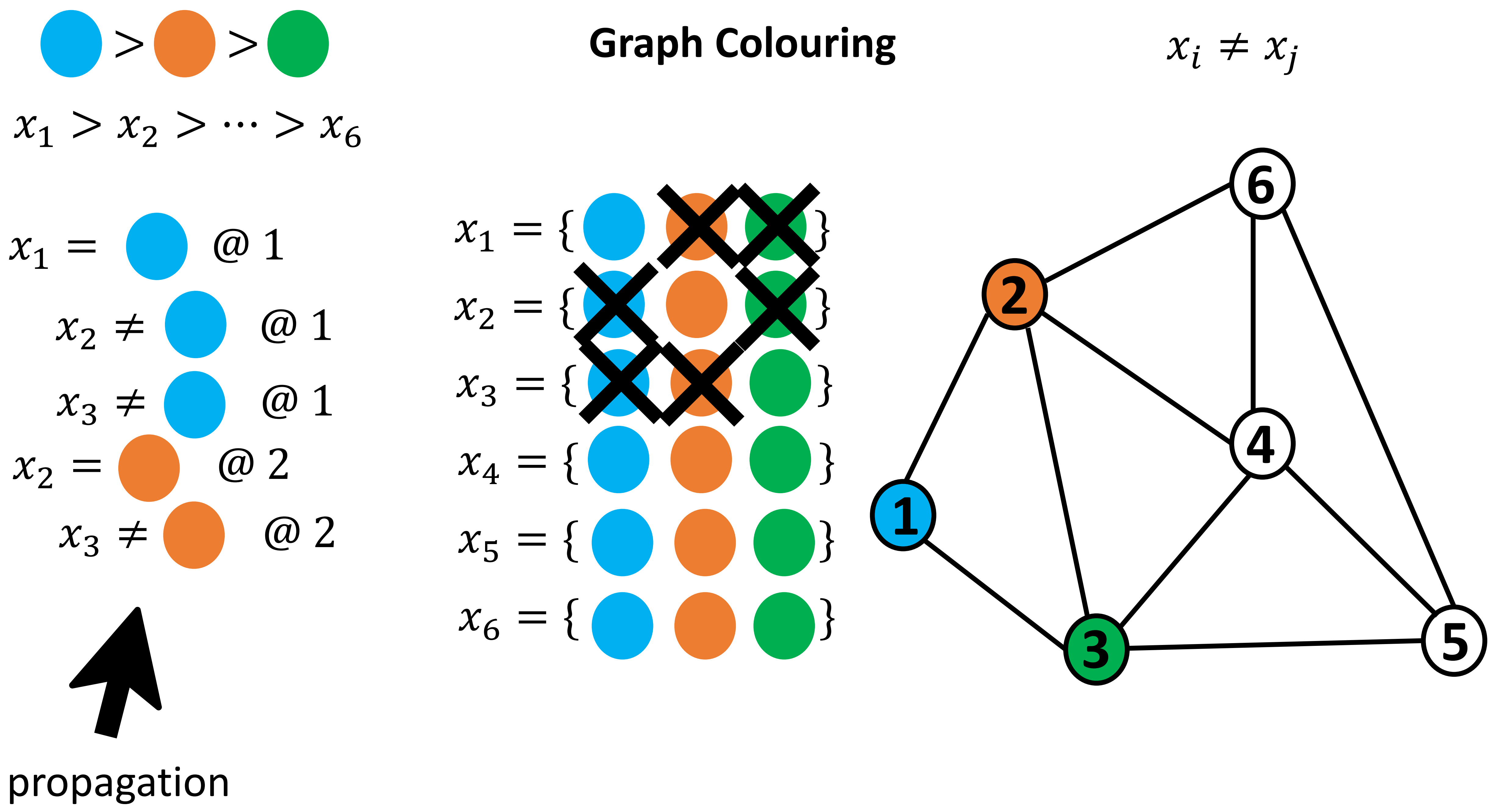


Graph Colouring
$x_i \neq x_j$
$x_1 > x_2 > \cdots > x_6$
decision
$x_1 =$ @ 1
$x_1 = \{ \}$
$x_2 = \{ \}$
$x_3 = \{ \}$
$x_4 = \{ \}$
$x_5 = \{ \}$
$x_6 = \{ \}$
1
2
3
4
5
6

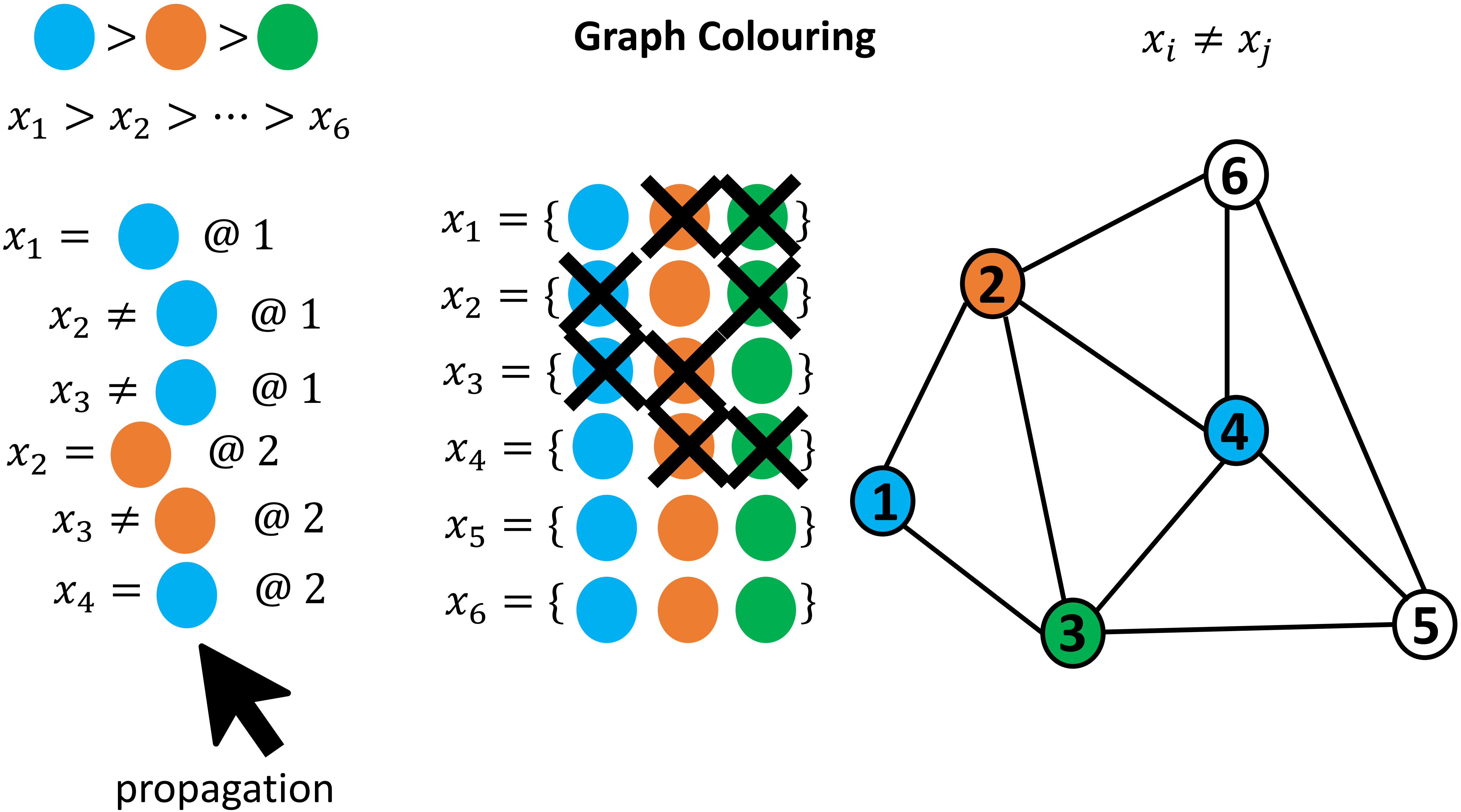

Graph Colouring
$x_i \neq x_j$
> >
$x_1 > x_2 > \cdots > x_6$
$x_1 =$ @ 1
$x_2 \neq$ @ 1
$x_3 \neq$ @ 1
propagations
$x_1 = \{$ $\}$
$x_2 = \{$ $\}$
$x_3 = \{$ $\}$
$x_4 = \{$ $\}$
$x_5 = \{$ $\}$
$x_6 = \{$ $\}$
1
2
3
4
5
6

Graph Colouring

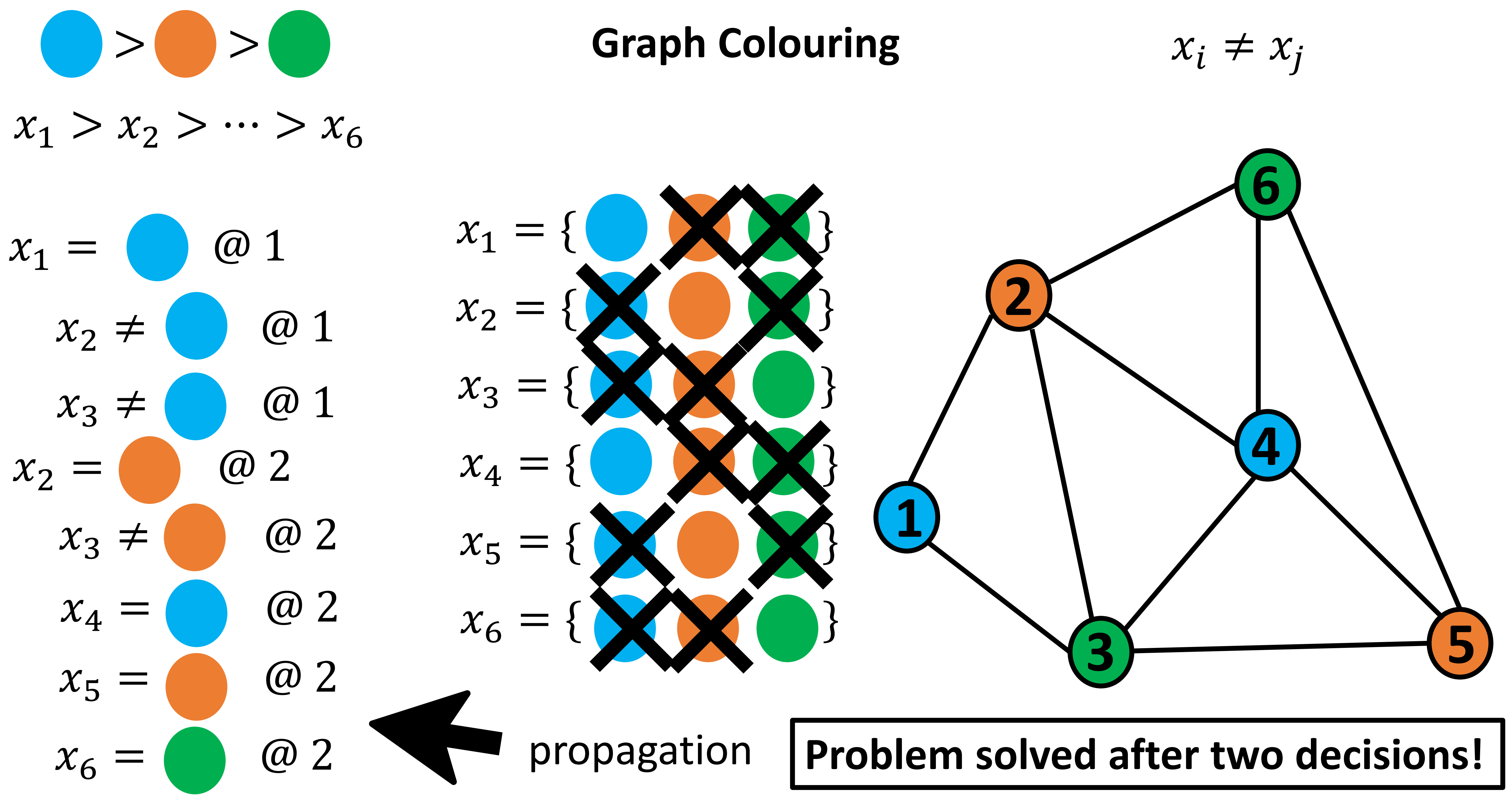




Graph Colouring
$x_i \neq x_j$
> >
$x_1 > x_2 > \cdots > x_6$
$x_1 =$ @ 1
$x_2 \neq$ @ 1
$x_3 \neq$ @ 1
$x_2 =$ @ 2
$x_3 \neq$ @ 2
propagation
$x_1 = \{ \}$
$x_2 = \{ \}$
$x_3 = \{ \}$
$x_4 = \{ \}$
$x_5 = \{ \}$
$x_6 = \{ \}$
1
2
3
4
5
6



Graph Colouring
$x_i \neq x_j$
$x_1 > x_2 > \cdots > x_6$
$x_1 =$ @ 1
$x_2 \neq$ @ 1
$x_3 \neq$ @ 1
$x_2 =$ @ 2
$x_3 \neq$ @ 2
$x_4 =$ @ 2
propagation
$x_1 = \{\ \}$
$x_2 = \{\ \}$
$x_3 = \{\ \}$
$x_4 = \{\ \}$
$x_5 = \{\ \}$
$x_6 = \{\ \}$
1
2
3
4
5
6



Graph Colouring
$x_i \neq x_j$
$x_1 > x_2 > \cdots > x_6$
$x_1 =$ @ 1
$x_2 \neq$ @ 1
$x_3 \neq$ @ 1
$x_2 =$ @ 2
$x_3 \neq$ @ 2
$x_4 =$ @ 2
$x_5 =$ @ 2
$x_6 =$ @ 2
$x_1 = \{ \}$
$x_2 = \{ \}$
$x_3 = \{ \}$
$x_4 = \{ \}$
$x_5 = \{ \}$
$x_6 = \{ \}$
propagation
1
2
3
4
5
6
Problem solved after two decisions!

Search Outline

Make a decision (branch)

Propagate based on constraints

Backtrack if conflict

Constraint Satisfaction Problem

Variables

Integer

$x_i \in \{0, 1, \dots, k\}$

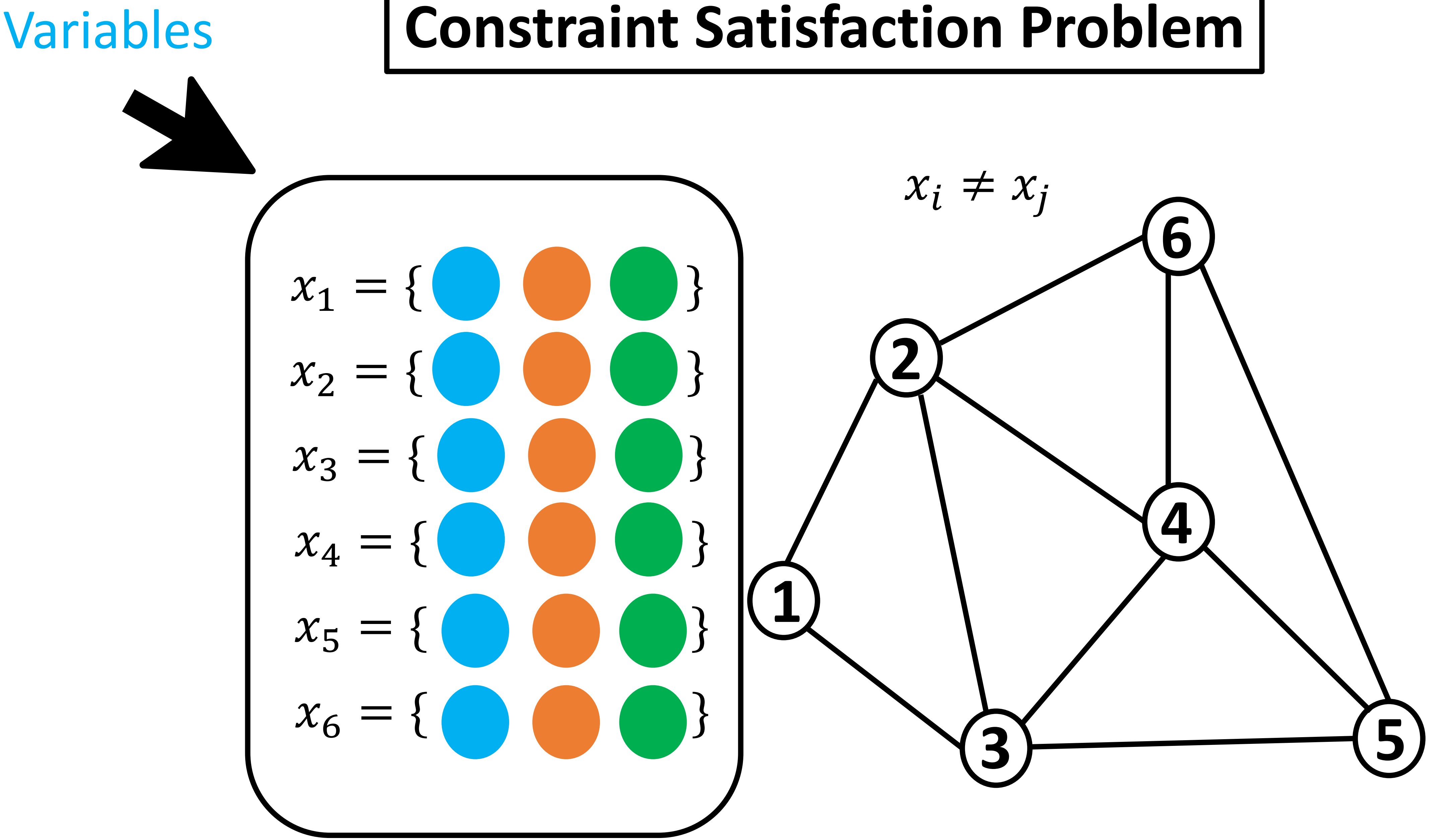
Constraints

Predicates

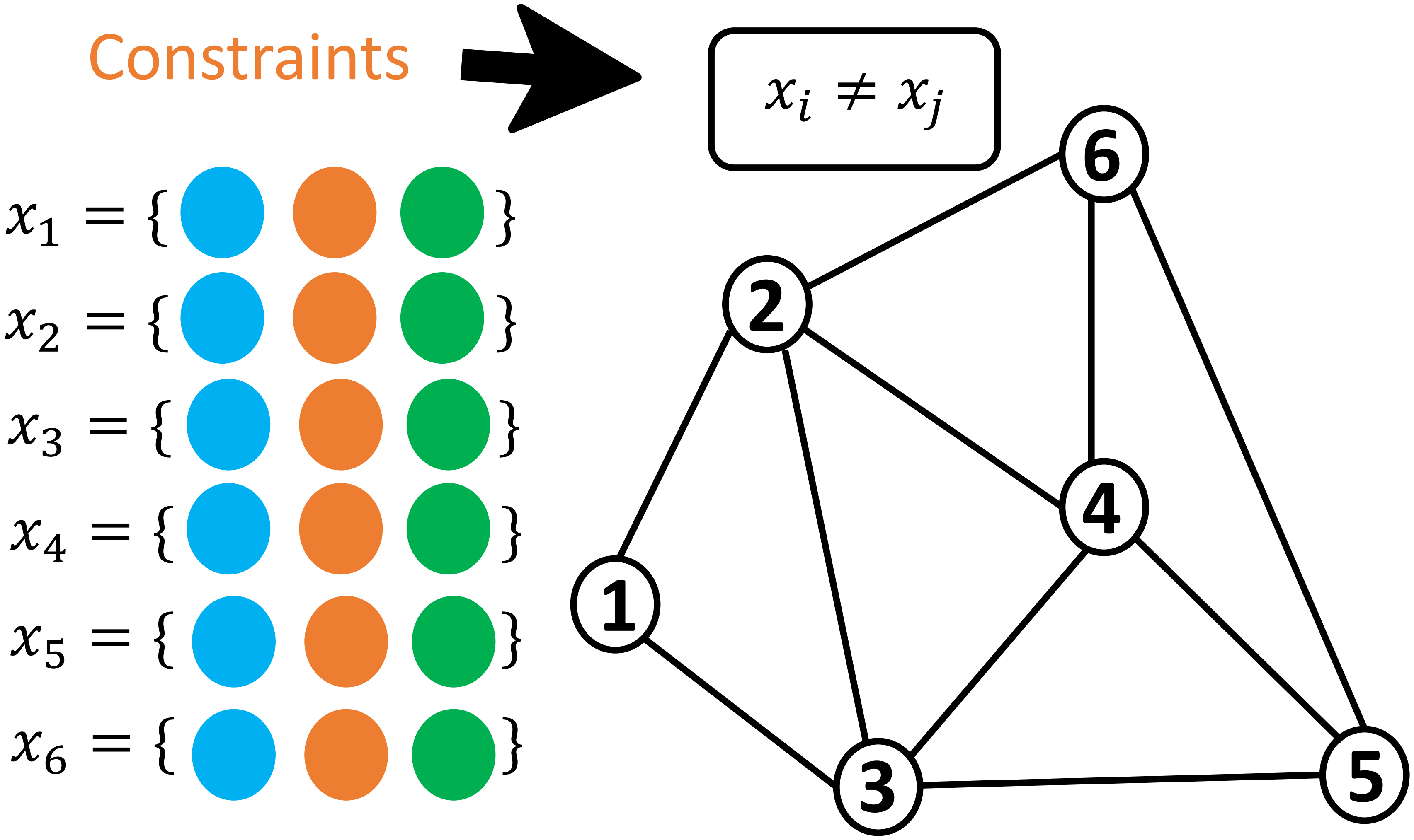
$C : Z^n \rightarrow \{\bot, \top\}$

$C(x_1, x_2, \dots, x_n)$

Constraint Satisfaction Problem



Constraint Satisfaction Problem



Constraint Satisfaction Problem

Variables

Constraints

Integer

$x_i \in \{0, 1, \ldots, 10\}$

Predicates

$C: X^n \rightarrow \{0,1\}$

$C(x_1, x_2, x_3)$

Graph Colouring

$v_i = \{\text{colours}\}$

$v_i \neq v_j$

Boolean/binary variable

$x \in \{0, 1\}$

Integer variable

$y \in \{0, 1, 2, \ldots, n\}$

Boolean/binary variable

$x \in \{0, 1\}$

Integer variable

$y \in \{0, 1, 2, \ldots, n\}$

domain

Set of values that the variable can be potentially assigned

Boolean/binary variable

$x \in \{0, 1\}$

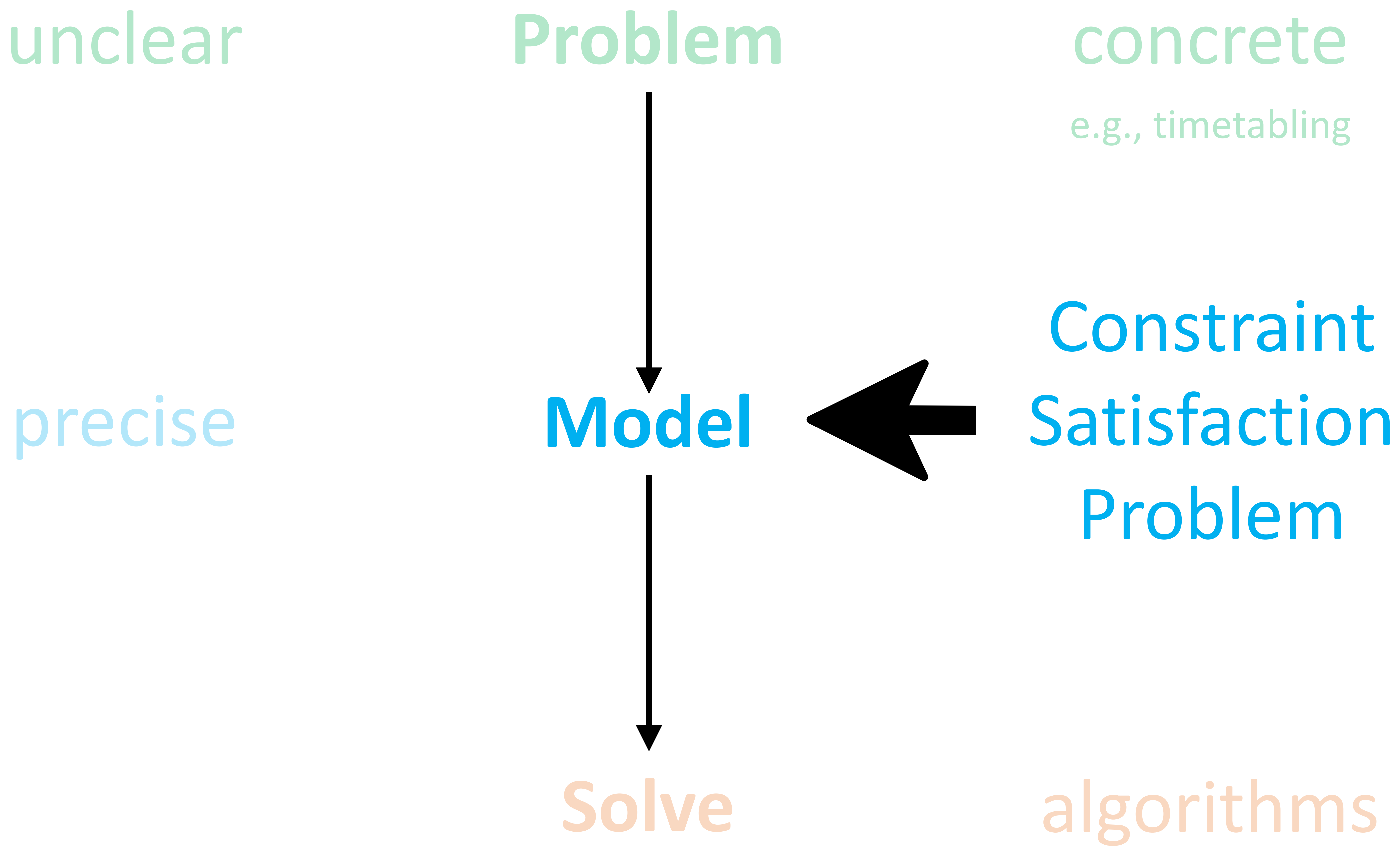
Integer variable

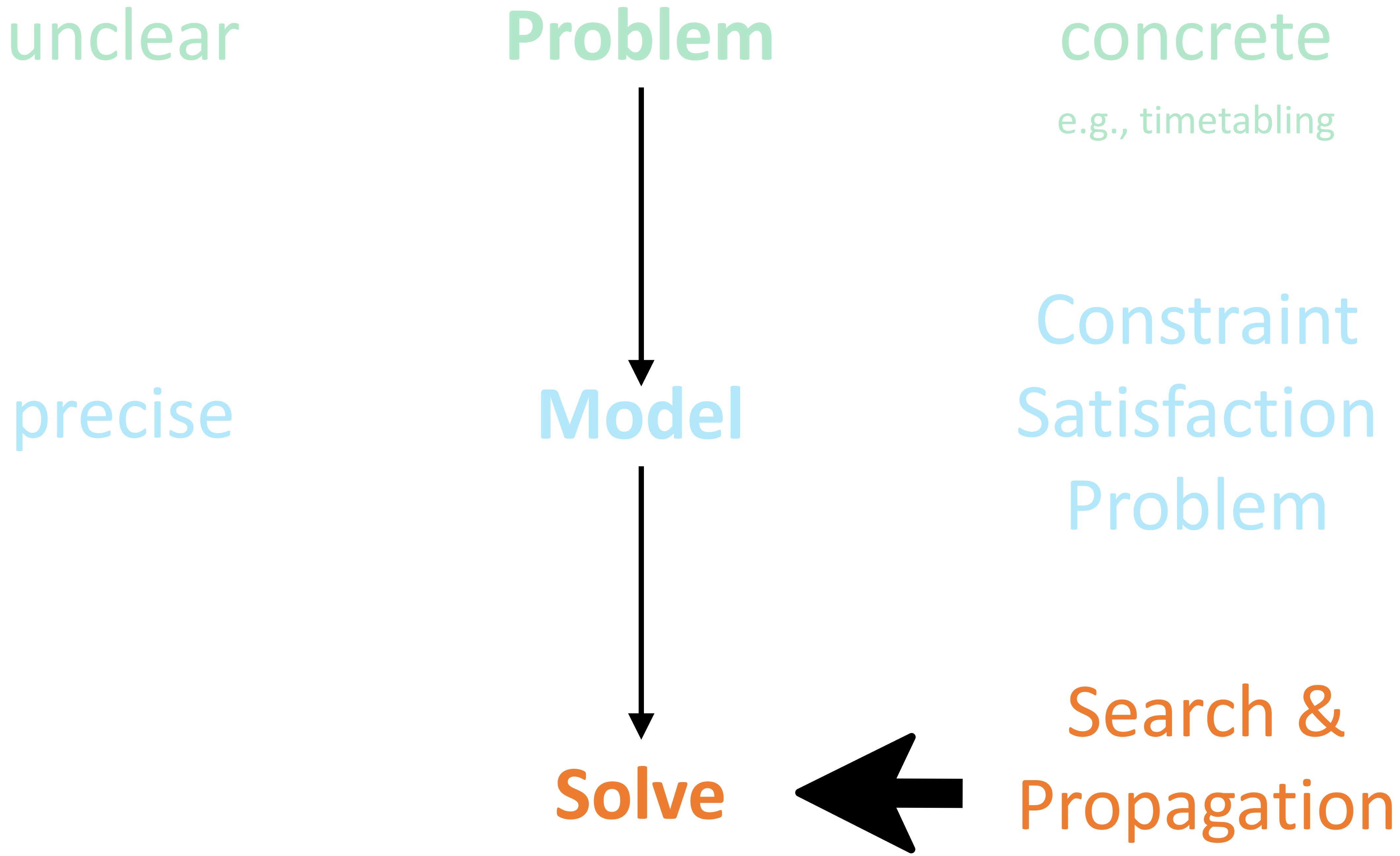
$y \in \{0, 1, 2, \ldots, n\}$

domain

Set of values that the variable can be potentially assigned

We consider finite domains!


unclear
Problem
concrete
e.g., timetabling
precise
Model
Constraint
Satisfaction
Problem
Solve
algorithms


unclear
Problem
concrete
e.g., timetabling
precise
Model
Constraint Satisfaction Problem
Solve
Search &
Propagation

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

Constraint $y_1 + 2y_2 \geq 4$

Variables

$y_1 \in \{0, 1, 2, 3, 4\}$

$y_2 \in \{0, 2\}$

Constraint

$$y_1 + 2y_2 \geq 4$$

Variables

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

Propagation

Shrink domains based on constraint

Constraint $y_1 + 2y_2 \geq 4$

Variables $y_1 \in \{0, 1, 2, 3, 4\}$

$y_2 \in \{0, 2\}$

Propagation **Shrink domains based on constraint**

$\rightarrow$ $f : D^n \rightarrow D^n$

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

Propagator

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\})$$

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\}) = (\{0, 1, 2, 3, 4\}, \{0, 2\})$$

No change: fixpoint

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\}) = (\{0, 1, 2, 3, 4\}, \{0, 2\})$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0\}) = \quad ?$$

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\}) = (\{0, 1, 2, 3, 4\}, \{0, 2\})$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0\}) = (\{4\}, \{0\})$$

Domain changed!

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\}) = (\{0, 1, 2, 3, 4\}, \{0, 2\})$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0\}) = (\{4\}, \{0\})$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, \cancel{4}\}, \{0\}) = \quad ?$$

$$y_1 + 2y_2 \geq 4$$

$$y_1 \in \{0, 1, 2, 3, 4\}$$

$$y_2 \in \{0, 2\}$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0, 2\}) = (\{0, 1, 2, 3, 4\}, \{0, 2\})$$

$$f(D_1, D_2) = f(\{0, 1, 2, 3, 4\}, \{0\}) = (\{4\}, \{0\})$$

Infeasible, conflict!

$$f(D_1, D_2) = f(\{0, 1, 2, 3, \cancel{4}\}, \{0\}) = (\{\}, \{\})$$

More complex example

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Propagate: remove values from domains!

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{\cancel{0}, \cancel{1}, \cancel{2}, 3, 4\}$$

How to propagate?

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

“Upper Bound”

$$y_2 = UB(y_2) = 2$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = ?$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_1) = ?$$

"Lower Bound"

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

"optimistic assignment"

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$

$$LB(y_1) = ?$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$

$$LB(y_1) = ?$$

$$y_1 + 2 \cdot 2 + 3 \cdot 4 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_1) = ?$$

$$y_1 + 2 \cdot 2 + 3 \cdot 4 \geq 17$$

$$y_1 + 16 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{0, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_1) = 1$$

$$y_1 + 2 \cdot 2 + 3 \cdot 4 \geq 17$$

$$y_1 + 16 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Value '0' removed by propagation

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_1) = 1$$

$$y_1 + 2 \cdot 2 + 3 \cdot 4 \geq 17$$
$$y_1 + 16 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Explanation of the propagation

$$\langle y_2 \leq 2 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_1 \geq 1 \rangle$$

$$y_2 = UB(y_2) = 2$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_1) = 1$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

$$y_1 = UB(y_2) = 4$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_2) = ?$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{0, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Value '0' removed by propagation

$$y_1 = UB(y_2) = 4$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_2) = 2$$

$$4 + 2 \cdot y_2 + 3 \cdot 4 \geq 17$$

$$y_2 + 16 \geq 17$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Explanation?

$$y_1 = UB(y_2) = 4$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_2) = 2$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{0, 1, 2, 3, 4\}$$

Explanation?

$$\langle y_1 \leq 4 \rangle \land \langle y_3 \leq 4 \rangle \Rightarrow \langle y_2 \geq 1 \rangle$$

$$y_1 = UB(y_2) = 4$$
$$y_3 = UB(y_3) = 4$$
$$LB(y_2) = 2$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{2\}$$

$$y_3 \in \{0, 1, 2, 3,4\}$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{1, 4\}$$

$$y_2 \in \{2\}$$

$$y_3 \in \{\cancel{0},\cancel{1},\cancel{2}, 3,4\}$$

Values '0', '1', '2' removed by propagation with explanation

$$\langle \boldsymbol{y_1 \leq 4} \rangle \wedge \langle \boldsymbol{y_2 \leq 2} \rangle \Rightarrow \langle \boldsymbol{y_3 \geq 3} \rangle$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{\cancel{0}, \cancel{1}, \cancel{2}, 3, 4\}$$

Propagation shrinks domains

$$f(\{0,1,4\},\{0,2\},\{0,1,2,3,4\}) = (\{1,4\},\{2\},\{3,4\})$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{\cancel{0}, \cancel{1}, \cancel{2}, 3, 4\}$$

Propagator as a function

$$f(\{0,1,4\},\{0,2\},\{0,1,2,3,4\}) = (\{1,4\},\{2\},\{3,4\})$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$y_1 \in \{\cancel{0}, 1, 4\}$$

$$y_2 \in \{\cancel{0}, 2\}$$

$$y_3 \in \{\cancel{0}, \cancel{1}, \cancel{2}, 3, 4\}$$

$$\langle y_2 \leq 2 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_1 \geq 1 \rangle$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_2 \geq 1 \rangle$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 2 \rangle \Rightarrow \langle y_3 \geq 3 \rangle$$

Propagation explained with three steps

$$\sum w_i \cdot y_i \geq k$$

(assume $w_i \geq 0$)

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

$$\sum w_i \cdot y_i \geq k$$

(assume $w_i \geq 0$)

set other variables to the optimistic assignment

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

$$\sum w_i \cdot y_i \geq k$$

(assume $w_i \geq 0$)

units that
must be covered by
the term $w_i \cdot y_i$

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

Think about the general case $w_i \in Z$

$$\sum w_i \cdot y_i \geq k$$

(assume $w_i \geq 0$)

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

Defines feasibility

Constraint

↓

Propagator

↓

Propagation Algorithm

Constraint

Propagator

Propagation
Algorithm

Defines rules
to shrink domains
without changing feasibility

Constraint

↓

Propagator

↓

Propagation Algorithm

Defines ***how*** the inference is computed

Data structures extremely important!

Search Outline

Make a decision (branch)

Propagate based on constraints

Backtrack if conflict

Our focus so far

Up next…

Checkers

More Propagators: Maximum

$$x = max(y, z)$$

$$\sum w_i \cdot y_i \geq k$$

In practice,
propagation algorithms can be complex…

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

$$\sum w_i \cdot y_i \geq k$$

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

In practice,
propagation algorithms can be complex…

Handle $\boldsymbol{w_i \in Z}$

$$\sum w_i \cdot y_i \geq k$$

In practice,
propagation algorithms can be complex...

Handle $\boldsymbol{w_i \in Z}$

Propagator

Incremental propagation

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

$$\sum w_i \cdot y_i \geq k$$

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

In practice,
propagation algorithms can be complex…

Handle $w_i \in Z$

Incremental propagation

Rounding errors

$$\sum w_i \cdot y_i \geq k$$

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

In practice,
propagation algorithms can be complex…

Handle $\boldsymbol{w_i \in Z}$

Incremental propagation

Rounding errors

Special cases, e.g., $\boldsymbol{w_i = 1}$

$$\sum w_i \cdot y_i \geq k$$

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

In practice, propagation algorithms can be complex…

Handle $\boldsymbol{w_i \in Z}$

Incremental propagation

Rounding errors

Special cases, e.g., $\boldsymbol{w_i = 1}$

→ **Many things can go wrong**

$$f(\{0,1,4\},\{0,2\},\{0,1,2,3,4\}) = (\{1,4\},\{2\},\{3,4\})$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

How to trust this propagation?

$$f(\{0,1,4\},\{0,2\},\{0,1,2,3,4\}) = (\{1,4\},\{2\},\{3,4\})$$

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_2 \leq 2 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_1 \geq 1 \rangle$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_2 \geq 1 \rangle$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 2 \rangle \Rightarrow \langle y_3 \geq 3 \rangle$$

Much simpler task:

Check the explanations!

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_2 \geq 1 \rangle$$

Is the explanation sound?

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \langle y_2 \geq 1 \rangle$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 0 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \perp$$

Reduce verification to feasibility checking

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 0 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \bot$$

Much simpler task!

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 0 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \perp$$

$$4 + 2 \cdot 0 + 3 \cdot 4 \geq 17$$

Assume optimistic assignment based on explanation

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 0 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \bot$$

$$4 + 2 \cdot 0 + 3 \cdot 4 \geq 17$$

$$16 \geq 17$$

Derive conflict to verify explanation!

$$y_1 + 2y_2 + 3y_3 \geq 17$$

$$\langle y_1 \leq 4 \rangle \wedge \langle y_2 \leq 0 \rangle \wedge \langle y_3 \leq 4 \rangle \Rightarrow \bot$$

$$4 + 2 \cdot 0 + 3 \cdot 4 \geq 17$$

$$16 \geq 17$$

Checker: verifies an explanation

1. Independent of the propagator

2. Much simpler logic → formally verified

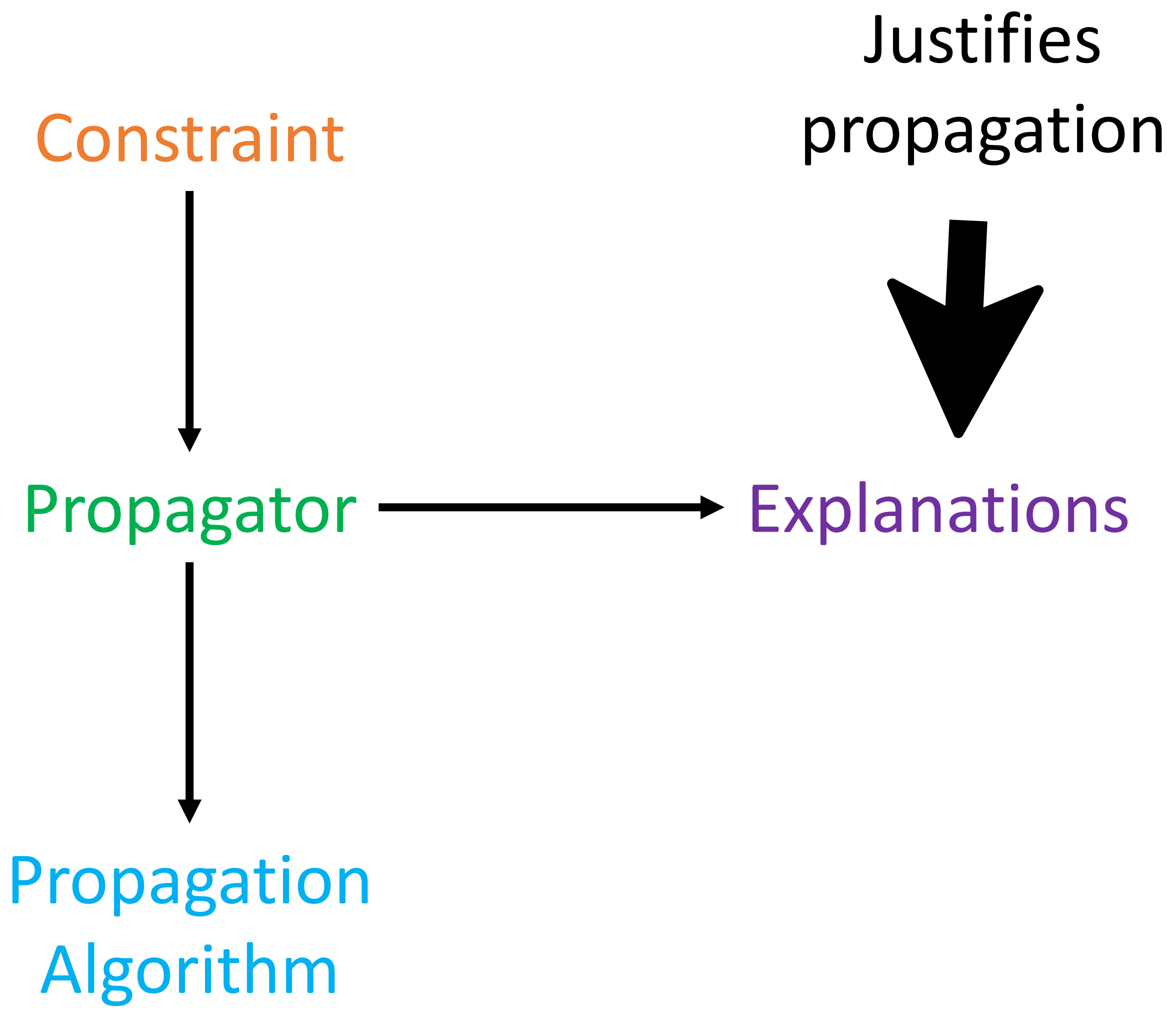
Constraint

↓

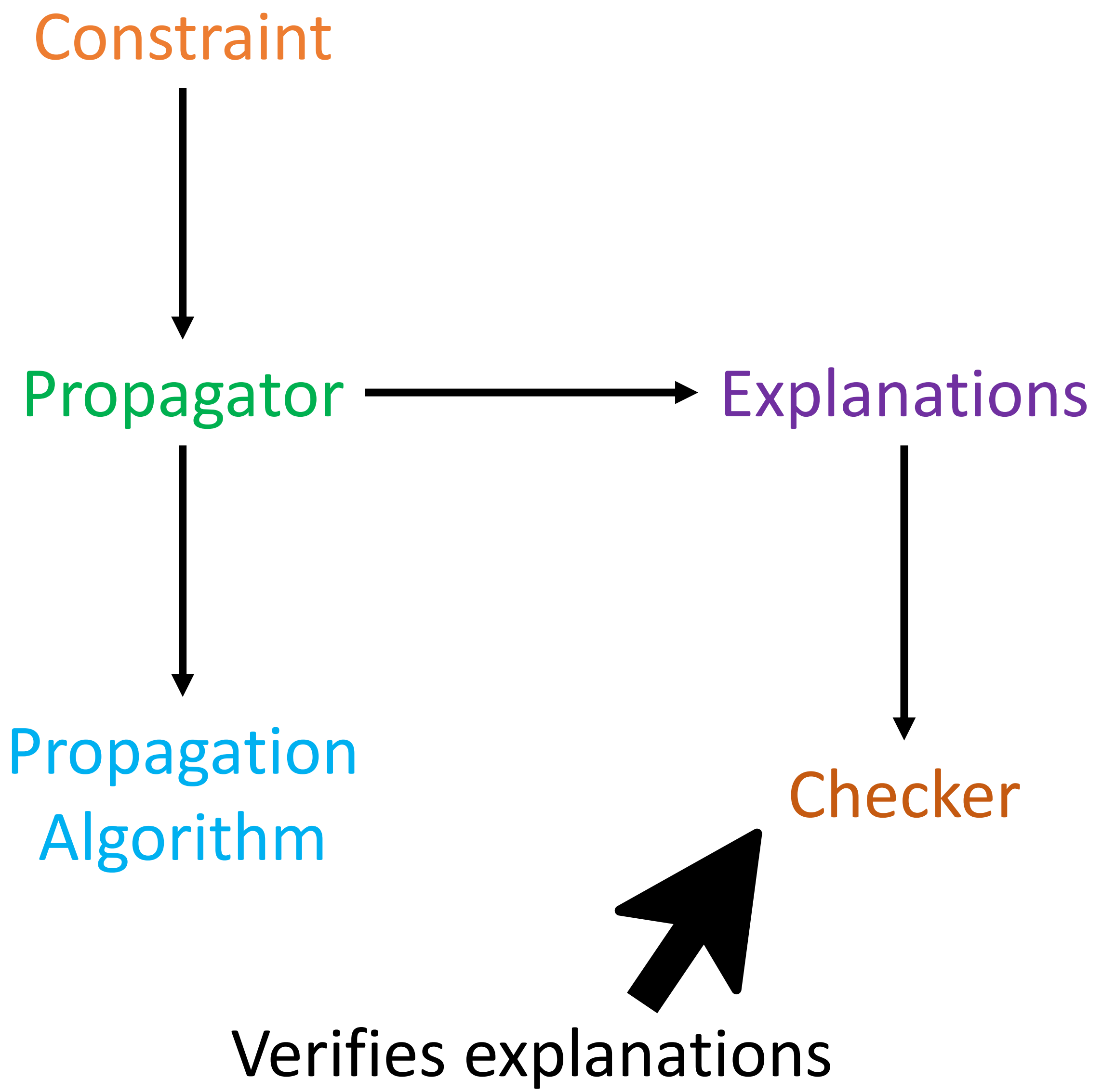
Propagator

↓

Propagation Algorithm



Justifies
propagation
Constraint
Propagator
Explanations
Propagation
Algorithm


Constraint
Propagator
Explanations
Propagation Algorithm
Checker
Verifies explanations

Propagation for the Maximum Constraint

$$x = \max(y, z)$$

Propagation for the Maximum Constraint

$$x = \max(y, z)$$

x, y, z are integer interval variables

Propagation for the Maximum Constraint

$$x = \max(y, z)$$

x, y, z are integer interval variables

Feasible solutions

$$x = 5 \quad y = 0 \quad z = 5$$

$$x = 10 \quad y = 10 \quad z = 10$$

$$x = 0 \quad y = -5 \quad z = 0$$

Propagation for the Maximum Constraint

$$x = \max(y, z)$$

x, y, z are integer interval variables

Feasible solutions

$x = 5$
$y = 0$
$z = 5$

$x = 10$
$y = 10$
$z = 10$

$x = 0$
$y = -5$
$z = 0$

Infeasible solutions

$x = 5$
$y = 10$
$z = 3$

$x = 10$
$y = 7$
$z = 9$

$x = 0$
$y = 1$
$z = 2$

Propagation for the Maximum Constraint

$$x = \max(y, z)$$

x, y, z are integer interval variables

Feasible solutions

$x = 5$
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$x = 0$
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$z = 2$

Devise the propagation rules!

Example

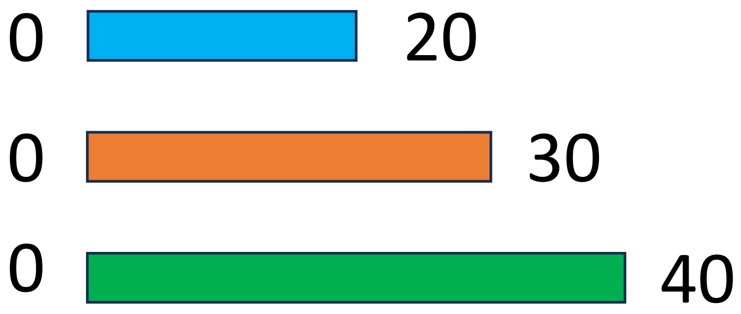
$$x = \max(y, z)$$

Before propagation

$x \in [0, 20]$

$y \in [0, 30]$

$z \in [0, 40]$



After propagation

Example

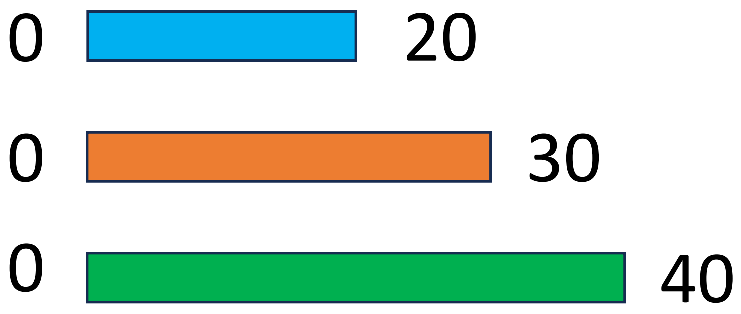
$$x = \max(y, z)$$

$\boldsymbol{x} \in [\mathbf{0}, \mathbf{20}]$

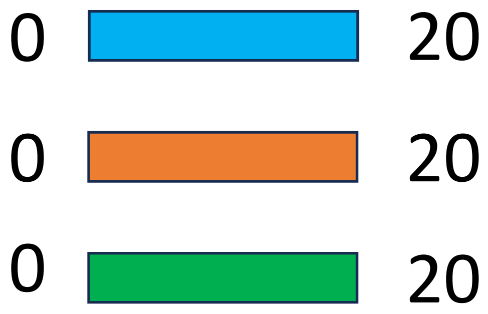
$\boldsymbol{y} \in [\mathbf{0}, \mathbf{30}]$

$\boldsymbol{z} \in [\mathbf{0}, \mathbf{40}]$

Before propagation



After propagation



Example

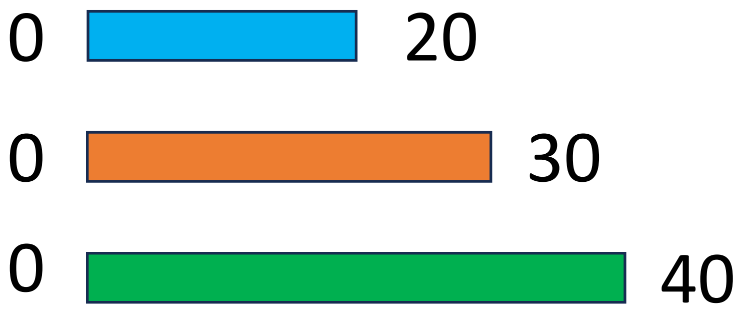
$$x = \max(y, z)$$

Before propagation

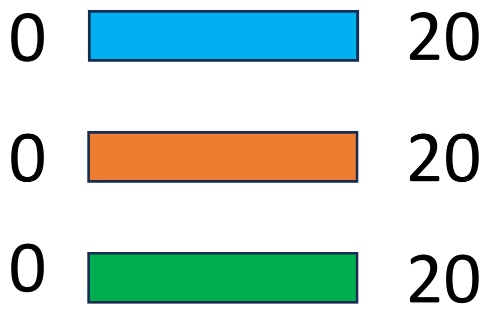
$x \in [0, 20]$

$y \in [0, 30]$

$z \in [0, 40]$



After propagation



Explanations

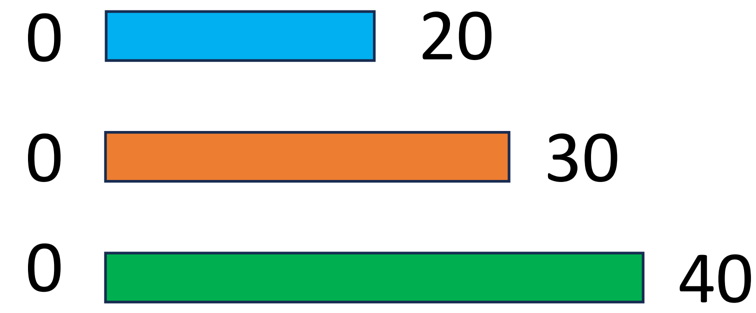
$\langle x \leq 20 \rangle \Rightarrow \langle y \leq 20 \rangle$

$\langle x \leq 20 \rangle \Rightarrow \langle z \leq 20 \rangle$

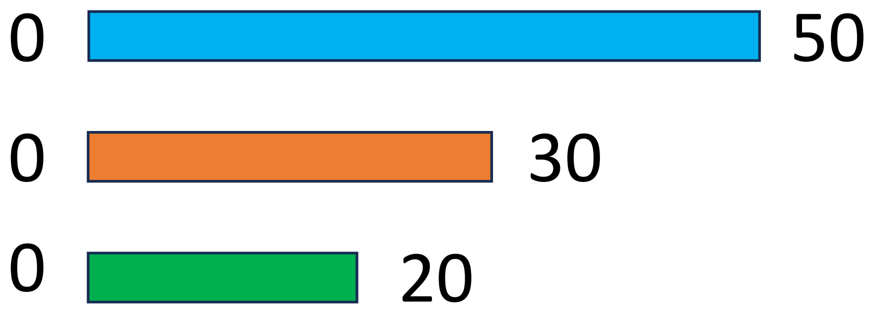
Devise the propagation rules!

$$x = \max(y, z)$$

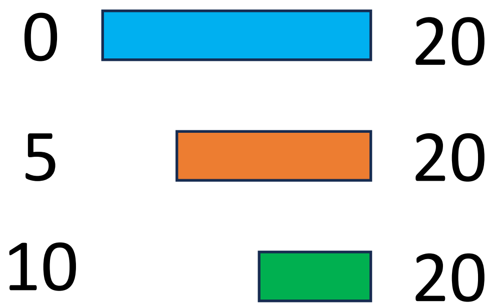
Scenario 1



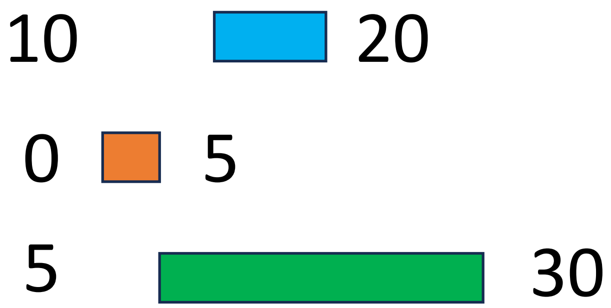
Scenario 3



Scenario 2

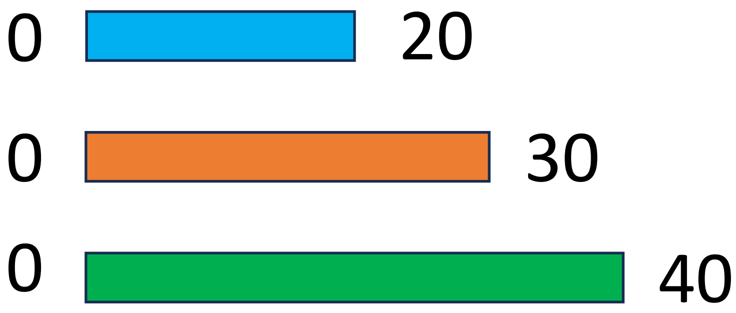


Scenario 4



$$x = \max(y, z)$$

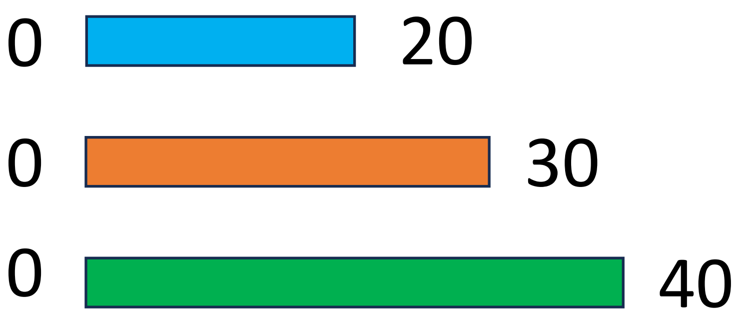
Scenario 1

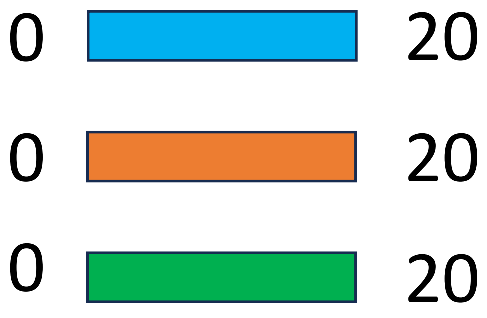


Rule?

$$x = \max(y, z)$$

Scenario 1



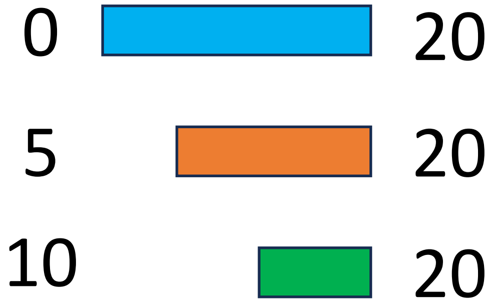


$$\langle x \leq k \rangle \Rightarrow \langle y \leq k \rangle$$

$$\langle x \leq k \rangle \Rightarrow \langle z \leq k \rangle$$

$$x = \max(y, z)$$

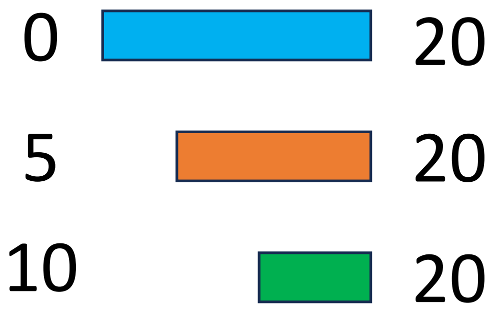
Scenario 2

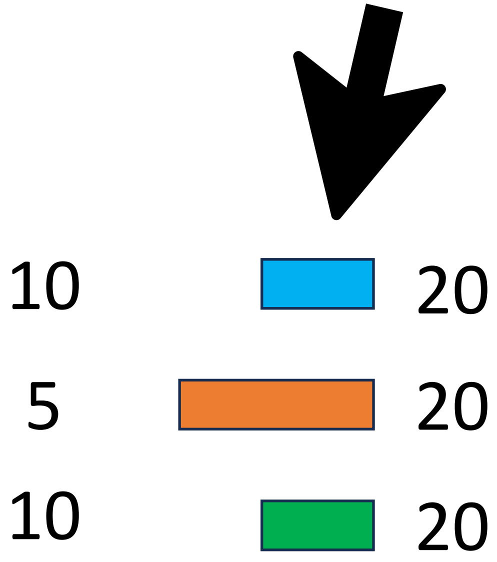


Propagation?

$$x = \max(y, z)$$

Scenario 2



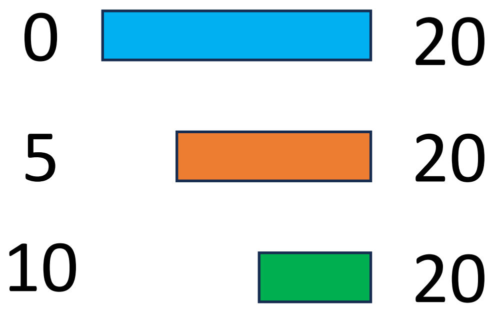


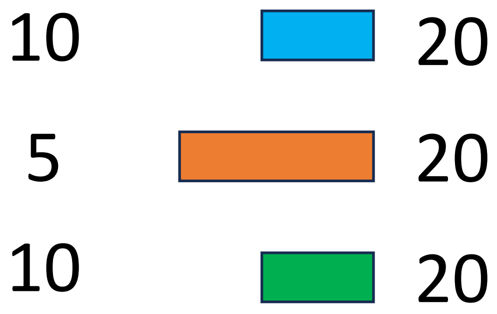
$$x = \max(y, z)$$

$$\langle y \geq k \rangle \Rightarrow \langle x \geq k \rangle$$

$$\langle z \geq k \rangle \Rightarrow \langle x \geq k \rangle$$

Scenario 2

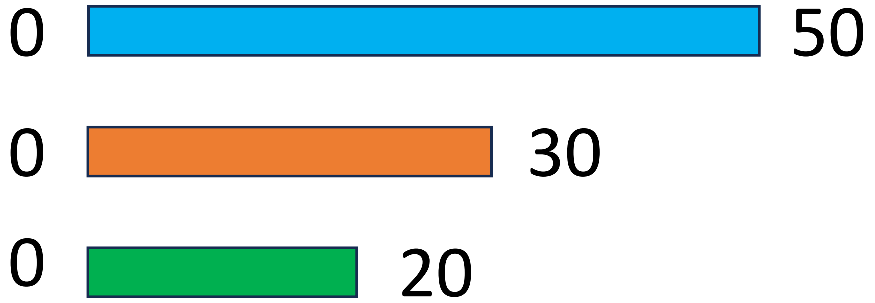




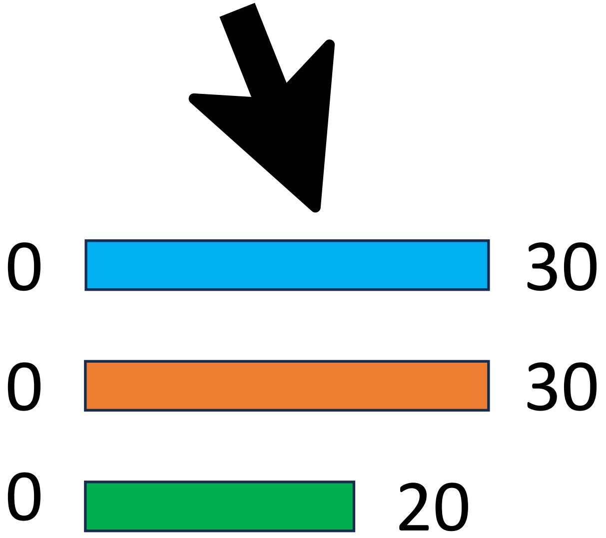
$$x = \max(y, z)$$

Propagation?

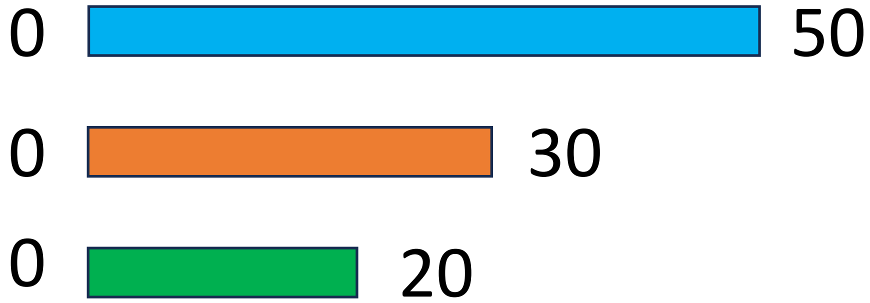
Scenario 3



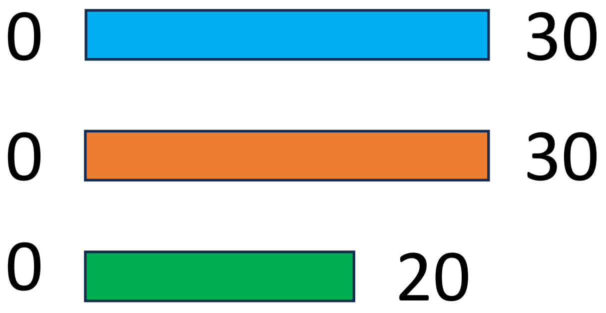
$$x = \max(y, z)$$



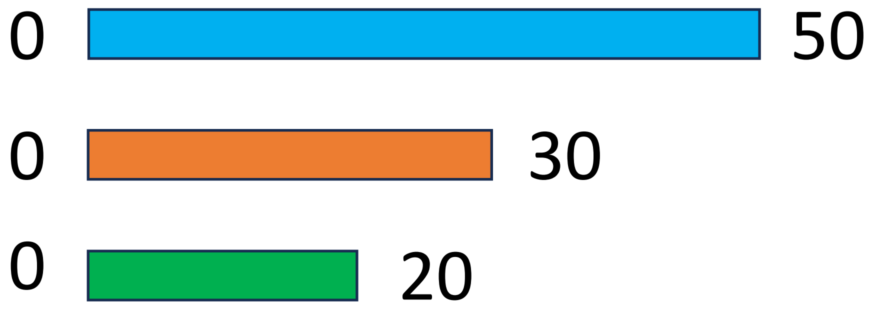
Scenario 3



$$x = \max(y, z)$$

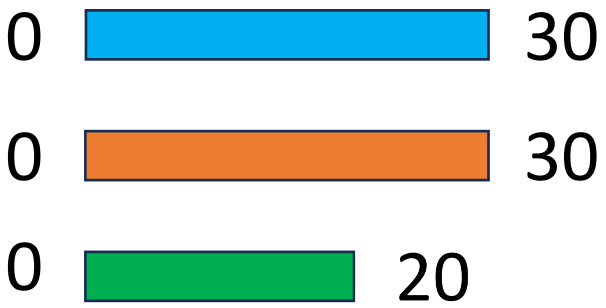


Scenario 3

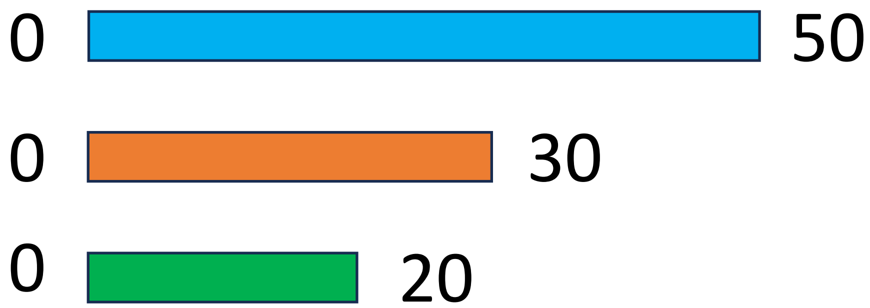


$$\langle y \leq a \rangle \wedge \langle z \leq b \rangle \Rightarrow \langle x \leq \max(a, b) \rangle$$

$$x = \max(y, z)$$



Scenario 3



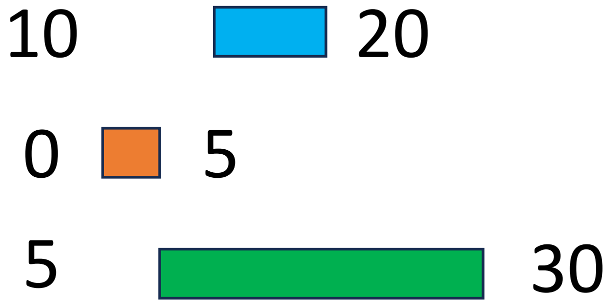
$$\langle y \leq a \rangle \wedge \langle z \leq b \rangle \Rightarrow \langle x \leq \max(a, b) \rangle$$

$$\langle y \leq k \rangle \wedge \langle z \leq k \rangle \Rightarrow \langle x \leq k \rangle$$

Lifted (more general) Explanation

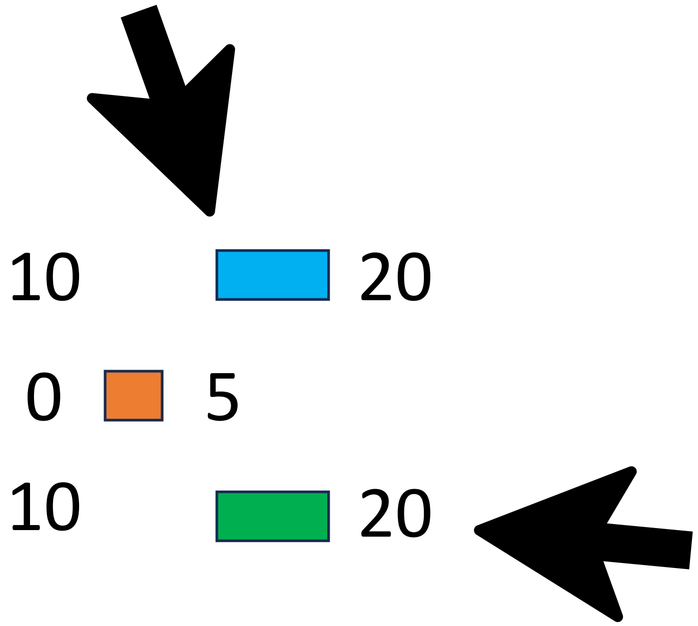
$$x = \max(y, z)$$

Scenario 4

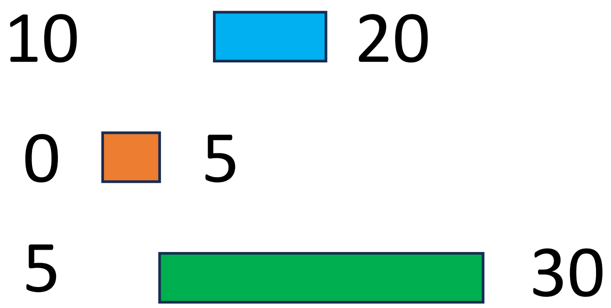


Propagation?

$$x = \max(y, z)$$

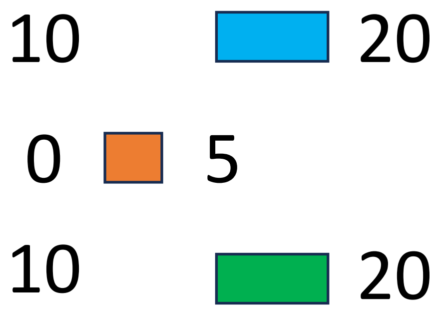


Scenario 4

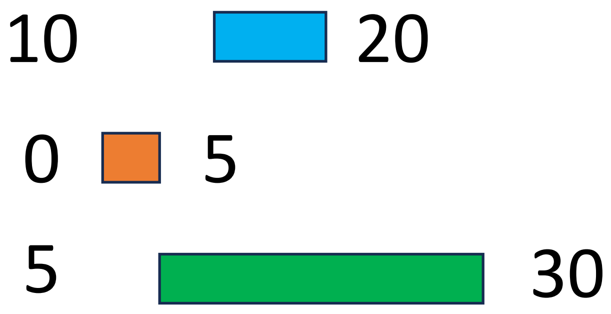


$$x = \max(y, z)$$

$$\langle x \geq a + 1 \rangle \wedge \langle y \leq a \rangle \Rightarrow \langle x = z \rangle$$



Scenario 4



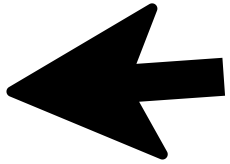
$$x = \max(y, z)$$

$$\langle x \le k \rangle \Rightarrow \langle y \le k \rangle$$

$$\langle x \le k \rangle \Rightarrow \langle z \le k \rangle$$

$$\langle y \le k \rangle \wedge \langle z \le k \rangle \Rightarrow \langle x \le k \rangle$$

Checker?



$$\langle y \ge k \rangle \Rightarrow \langle x \ge k \rangle$$

$$\langle z \ge k \rangle \Rightarrow \langle x \ge k \rangle$$

$$\langle x \ge a + 1 \rangle \wedge \langle y \le a \rangle \Rightarrow \langle x = z \rangle$$

$$x = \max(y, z)$$

$$\langle x \le k \rangle \Rightarrow \langle y \le k \rangle$$

$$\langle x \le k \rangle \Rightarrow \langle z \le k \rangle$$

$$\langle y \le k \rangle \land \langle z \le k \rangle \Rightarrow \langle x \le k \rangle$$

Only one checker rule → homework!

$$\langle y \ge k \rangle \Rightarrow \langle x \ge k \rangle$$

$$\langle z \ge k \rangle \Rightarrow \langle x \ge k \rangle$$

$$\langle x \ge a + 1 \rangle \land \langle y \le a \rangle \Rightarrow \langle x = z \rangle$$

Summary

Constraint Programming

Search

Constraints and Propagators

Linear inequality: $\sum w_i x_i \geq c$

Max: $x = max(y, z)$

Checkers

Next time…

Constraints and Propagators

All-Different

Cumulative

Checker for Max → homework!