Propagation: AllDifferent and Cumulative

Modern Constraint Programming

(ESSAI'26)

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Last time…

Constraint Programming

Search

Constraints and Propagators

Linear inequality: $\sum w_i x_i \geq c$
Max: $x = max(y, z)$

Checkers

Combinatorial Optimisation Problem

$$\boldsymbol{min\ F(X)}$$
$$\boldsymbol{X \in C \subseteq \mathbb{N}^{n}}$$

Search Outline

Make a decision (branch)

Propagate based on constraints

Backtrack if conflict

$$\sum w_i \cdot y_i \geq k$$

(assume $w_i \geq 0$)

Propagator

$$y_i \geq \left\lceil \frac{k - \sum_{j \neq i} w_j \cdot UB(y_j)}{w_i} \right\rceil$$

$$x = \max(y, z)$$

$\langle x \le k \rangle \Rightarrow \langle y \le k \rangle$

$\langle x \le k \rangle \Rightarrow \langle z \le k \rangle$

$\langle y \le k \rangle \wedge \langle z \le k \rangle \Rightarrow \langle x \le k \rangle$

Checker?

$\langle y \ge k \rangle \Rightarrow \langle x \ge k \rangle$

$\langle z \ge k \rangle \Rightarrow \langle x \ge k \rangle$

$\langle x \ge a + 1 \rangle \wedge \langle y \le a \rangle \Rightarrow \langle x = z \rangle$

In this lecture…

Constraints and Propagators

All-Different

Cumulative

Up next…

All-Different

Decomposition

Main idea for checker and propagation

AllDifferent Constraint

“all variables must take distinct values”

All-Different(x_1, x_2, x_3, x_4)

AllDifferent Constraint

"all variables must take distinct values"

All-Different(x_1, x_2, x_3, x_4)

➔ **Common in assignment problems!**

AllDifferent Constraint

"all variables must take distinct values"

All-Different(x_1, x_2, x_3, x_4)

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}}$$

AllDifferent Constraint

"all variables must take distinct values"

All-Different(x_1, x_2, x_3, x_4)

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 5 \\ x_4 &= 3 \end{aligned}$$

AllDifferent Constraint

$$x_i \in D_i \subset \mathbb{N}$$

$$\text{All-Different}(x_1, x_2, \dots, x_n)$$

How to propagate?

$$x_1, x_2, x_3, x_4 \in \{1, 2, 3, 4\}$$

"all variables must take distinct values"

$$x_i \neq x_j$$

$$\boxed{\begin{array}{lll} x_1 \neq x_2 & & \\ x_1 \neq x_3 & x_2 \neq x_3 & \\ x_1 \neq x_4 & x_2 \neq x_4 & x_3 \neq x_4 \end{array}}$$

$x_1 \in \{1, 2, 3, 4\}$
$x_2 \in \{1, 2, 3, 4\}$
$x_3 \in \{1, 2, 3, 4\}$
$x_4 \in \{1, 2, 3, 4\}$

No propagation

$x_1 \neq x_2$

$x_1 \neq x_3$ $x_2 \neq x_3$

$x_1 \neq x_4$ $x_2 \neq x_4$ $x_3 \neq x_4$

$x_1 \in \{1, 2, 3, 4\}$
$x_2 \in \{1, 2, 3, 4\}$
$x_3 \in \{1, 2, 3, 4\}$
$x_4 \in \{1\}$

Propagate?

$x_1 \neq x_2$

$x_1 \neq x_3 \qquad x_2 \neq x_3$

$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$

$x_1 \in \{\cancel{1}, 2, 3, 4\}$

$x_2 \in \{1, 2, 3, 4\}$

$x_3 \in \{1, 2, 3, 4\}$

$x_4 \in \{1\}$

$\langle x_4 = 1 \rangle \Rightarrow \langle x_1 \neq 1 \rangle$

$x_1 \neq x_2$

$x_1 \neq x_3 \quad x_2 \neq x_3$

$x_1 \neq x_4 \quad x_2 \neq x_4 \quad x_3 \neq x_4$

$$x_1 \in \{\cancel{1}, 2, 3, 4\}$$
$$x_2 \in \{\cancel{1}, 2, 3, 4\}$$
$$x_3 \in \{\cancel{1}, 2, 3, 4\}$$
$$x_4 \in \{1\}$$

$$\langle x_i = k \rangle \Rightarrow \langle x_j \neq k \rangle$$

$$x_1 \neq x_2$$

$$x_1 \neq x_3 \qquad x_2 \neq x_3$$

$$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$$

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{1, 2, 4\}$$
$$x_4 \in \{1, 2\}$$

Propagate?

$x_1 \neq x_2$

$x_1 \neq x_3$ $\quad$ $x_2 \neq x_3$

$x_1 \neq x_4$ $\quad$ $x_2 \neq x_4$ $\quad$ $x_3 \neq x_4$

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{1, 2, 4\}$$
$$x_4 \in \{1, 2\}$$

$$\langle x_i = k \rangle \Rightarrow \langle x_j \neq k \rangle$$

No propagation!

$x_1 \neq x_2$		
$x_1 \neq x_3$	$x_2 \neq x_3$	
$x_1 \neq x_4$	$x_2 \neq x_4$	$x_3 \neq x_4$

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{1, 2, 4\}$$
$$x_4 \in \{1, 2\}$$

'4' must be assigned here

$x_1 \neq x_2$

$x_1 \neq x_3$ $\quad x_2 \neq x_3$

$x_1 \neq x_4$ $\quad x_2 \neq x_4$ $\quad x_3 \neq x_4$

$x_1 \in \{1, 3\}$
$x_2 \in \{1, 3\}$
$x_3 \in \{4\}$
$x_4 \in \{1, 2\}$ ← **'2' must be assigned here**

$$x_1 \neq x_2$$

$$x_1 \neq x_3 \qquad x_2 \neq x_3$$

$$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$$

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{4\}$$
$$x_4 \in \{2\}$$

$$x_1 \neq x_2$$

$$x_1 \neq x_3 \qquad x_2 \neq x_3$$

$$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$$

$$x_1 \in \{1,3\}$$
$$x_2 \in \{1,3\}$$
$$x_3 \in \{4\}$$
$$x_4 \in \{2\}$$

Stronger reasoning by combining constraints!

$x_1 \neq x_2$

$x_1 \neq x_3$ $\quad$ $x_2 \neq x_3$

$x_1 \neq x_4$ $\quad$ $x_2 \neq x_4$ $\quad$ $x_3 \neq x_4$

$$x_1 \in \{1, 2, 3\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{1, 2, 3\}$$
$$x_4 \in \{1, 2\}$$

$$x_1 \neq x_2$$

$$x_1 \neq x_3 \qquad x_2 \neq x_3$$

$$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$$

$$x_1 \in \{1, 2, 3\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{1, 2, 3\}$$
$$x_4 \in \{1, 2\}$$

no propagation
even though
infeasible!

$$\boxed{\begin{array}{lll} x_1 \neq x_2 & & \\ x_1 \neq x_3 & x_2 \neq x_3 & \\ x_1 \neq x_4 & x_2 \neq x_4 & x_3 \neq x_4 \end{array}}$$

Variant 1:
Decomposition
for All-Different

All-Different, Version 1: Decomposition

$$x_1, x_2, x_3, x_4 \in \{1, 2, 3, 4\}$$

"all variables must take distinct values"

$$x_i \neq x_j$$

Pros?

$$\boxed{\begin{array}{lll} x_1 \neq x_2 & & \\ x_1 \neq x_3 & x_2 \neq x_3 & \\ x_1 \neq x_4 & x_2 \neq x_4 & x_3 \neq x_4 \end{array}}$$

Cons?

All-Different, Version 1: Decomposition

$$x_1, x_2, x_3, x_4 \in \{1, 2, 3, 4\}$$

"all variables must take distinct values"

$$x_i \neq x_j$$

Computationally inexpensive

$$x_1 \neq x_2$$

$$x_1 \neq x_3 \qquad x_2 \neq x_3$$

$$x_1 \neq x_4 \qquad x_2 \neq x_4 \qquad x_3 \neq x_4$$

Misses propagations

Variant 2:
Domain Consistency
for All-Different

All-Different: Main idea for stronger propagation

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{1, 2, 3, 4, 5, 6\}$$
$$x_4 \in \{1, 2\}$$

All-Different: Main idea for stronger propagation

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{1, 2, 3, 4, 5, 6\}$$
$$x_4 \in \{1, 2\}$$

All-Different: Main idea for stronger propagation

$$x_1 \in \{1,3\}$$
$$x_2 \in \{1,3\}$$
$$x_3 \in \{1,2,3,4,5,6\}$$
$$x_4 \in \{1,2\}$$

$$H = \{x_1, x_2, x_3\}$$

$$Union\ of\ domains\ in\ H = \{1,2,3\}$$

What can we conclude about x_3?

All-Different: Main idea for stronger propagation

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{\cancel{1}, \cancel{2}, \cancel{3}, 4, 5, 6\}$$
$$x_4 \in \{1, 2\}$$

$$H = \{x_1, x_2, x_3\}$$

$$Union\ of\ domains\ in\ H = \{1, 2, 3\} \quad \rightarrow \quad x_3 \neq \{1, 2, 3\}$$

(variables in H are occupying values {1, 2, 3}, so x_3 cannot take any of those values)

All-Different: Main idea for stronger propagation

$x_1 \in \{1, 3\}$

$x_2 \in \{1, 3\}$

$x_3 \in \{\cancel{1}, \cancel{2}, \cancel{3}, 4, 5, 6\}$

$x_4 \in \{1, 2\}$

$H = \{x_1, x_2\}$

$Union\ of\ domains\ in\ H = \{1, 3\}$

What can we conclude about x_4?

All-Different: Main idea for stronger propagation

$$x_1 \in \{1, 3\}$$
$$x_2 \in \{1, 3\}$$
$$x_3 \in \{\cancel{1}, \cancel{2}, \cancel{3}, 4, 5, 6\}$$
$$x_4 \in \{\cancel{1}, 2\}$$

$$H = \{x_1, x_2\}$$

$$\mathit{Union\ of\ domains\ in\ H} = \{1, 3\} \quad \rightarrow \quad x_4 \neq 1$$

Another Example: Propagate?

$$x_1 \in \{1, 2, 3\}$$

$$x_2 \in \{1, 2, 3\}$$

$$x_3 \in \{1, 2, 3\}$$

$$x_4 \in \{1, 2, 3\}$$

Hall set

$$x_1 \in \{1, 2, 3\}$$

$$x_2 \in \{1, 2, 3\}$$

$$x_3 \in \{1, 2, 3\}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{3}\}$$

$$x_1 \in \{1, 2, 3\}$$

$$x_2 \in \{1, 2, 3\}$$

$$x_3 \in \{1, 2, 3\}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{3}\}$$

Conflict explanation?

$$
\begin{gathered}
\langle x_1 \geq 1\rangle \land \langle x_1 \leq 3\rangle \land \\
\langle x_2 \geq 1\rangle \land \langle x_2 \leq 3\rangle \land \\
\langle x_3 \geq 1\rangle \land \langle x_3 \leq 3\rangle \land \\
\langle x_4 \geq 1\rangle \land \langle x_4 \leq 3\rangle \\
\Rightarrow \\
\bot
\end{gathered}
$$

$$\boxed{\begin{gathered} x_1 \in \{1, 2, 3\} \\ x_2 \in \{1, 2, 3\} \\ x_3 \in \{1, 2, 3\} \end{gathered}}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{3}\}$$

Example with Holes

Hall set

$$x_1 \in \{1, 2, 4\}$$

$$x_2 \in \{1, 2, 4\}$$

$$x_3 \in \{1, 2, 4\}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{4}\}$$

Example with Holes

$$\langle x_1 \geq 1 \rangle \wedge \langle x_1 \neq 1 \rangle \wedge \langle x_1 \leq 4 \rangle \ \wedge$$
$$\langle x_2 \geq 1 \rangle \wedge \langle x_2 \neq 1 \rangle \wedge \langle x_2 \leq 4 \rangle \ \wedge$$
$$\langle x_3 \geq 1 \rangle \wedge \langle x_3 \neq 1 \rangle \wedge \langle x_3 \leq 4 \rangle \wedge$$
$$\langle x_4 \geq 1 \rangle \wedge \langle x_4 \neq 1 \rangle \wedge \langle x_4 \leq 4 \rangle$$
$$\Rightarrow$$
$$\perp$$

$$\boxed{\begin{array}{l} x_1 \in \{1, 2, 4\} \\ x_2 \in \{1, 2, 4\} \\ x_3 \in \{1, 2, 4\} \end{array}}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{4}\}$$

Hall set

n variables whose domains cover exactly n values

Hall set

n variables whose domains cover exactly n values

$$x_1 \in \{1, 2, 4\}$$

$$x_2 \in \{1, 2, 4\}$$

$$x_3 \in \{1, 2, 4\}$$

$$H = \{x_1, x_2, x_3\}$$

$$\bigcup D(x_i) = \{1, 2, 4\}$$

$$x_4 \in \{\cancel{1}, \cancel{2}, \cancel{4}, 5, 6, 7\}$$

Hall set

n variables whose domains cover exactly n values

Every conflict and propagation is explained by a Hall set

Hall set

n variables whose domains cover exactly n values

$\boxed{\text{Every conflict and propagation is explained by a Hall set}}$

$\downarrow$

Easy to design a checker!

Valid explanation?

$$\langle x_1 \geq 1\rangle \wedge \langle x_1 \leq 2\rangle \wedge$$
$$\langle x_2 \geq 1\rangle \wedge \langle x_2 \leq 2\rangle \wedge$$
$$\langle x_3 \geq 1\rangle \wedge \langle x_3 \leq 2\rangle$$
$$\Rightarrow$$
$$\perp$$

Reconstruct domains

$$x_1 \in \{1, 2\}$$

$$x_2 \in \{1, 2\}$$

$$x_3 \in \{1, 2\}$$

$$\langle x_1 \geq 1 \rangle \wedge \langle x_1 \leq 2 \rangle \ \wedge$$
$$\langle x_2 \geq 1 \rangle \wedge \langle x_2 \leq 2 \rangle \ \wedge$$
$$\langle x_3 \geq 1 \rangle \wedge \langle x_3 \leq 2 \rangle$$
$$\Rightarrow$$
$$\perp$$

$$\langle x_1 \geq 1\rangle \wedge \langle x_1 \leq 2\rangle \wedge$$
$$\langle x_2 \geq 1\rangle \wedge \langle x_2 \leq 2\rangle \wedge$$
$$\langle x_3 \geq 1\rangle \wedge \langle x_3 \leq 2\rangle$$
$$\Rightarrow$$
$$\perp$$

Reconstruct domains

$$x_1 \in \{1, 2\}$$
$$x_2 \in \{1, 2\}$$
$$x_3 \in \{1, 2\}$$

Check condition

$$|\{1, 2\}| < |\{x_1, x_2, x_3\}|$$

✓

Explanation verified

Not a valid explanation!

$$\langle x_1 \geq 1\rangle \wedge \langle x_1 \leq 3\rangle \wedge$$
$$\langle x_2 \geq 1\rangle \wedge \langle x_2 \leq 2\rangle \wedge$$
$$\langle x_3 \geq 1\rangle \wedge \langle x_3 \leq 2\rangle$$
$$\Rightarrow$$
$$\bot$$

Reconstruct domains

$$x_1 \in \{1, 2, 3\}$$
$$x_2 \in \{1, 2\}$$
$$x_3 \in \{1, 2\}$$

Check condition

$$|\{1, 2, 3\}| < |\{x_1, x_2, x_3\}|$$

$\times$

Explanation rejected

Checker Procedure

1. Reconstruct domains based on the explanation

2. Check condition

$$H = \{x_i\}$$

$$\bigcup D(x_i) < |H|\,?$$

Checker Procedure

Easy to verify correctness of explanations!

1. Reconstruct domains based on the explanation

(even without knowing
the conflict/propagation algorithm)

2. Check condition

$$H = \{x_i\}$$

$$\bigcup D(x_i) < |H|\,?$$

Another Example: Propagate?

$$x_1 \in \{1, 2\}$$

$$x_2 \in \{1, 2, 3\}$$

$$x_3 \in \{3, 4\}$$

$$x_4 \in \{3, 4, 5\}$$

$$x_5 \in \{4, 5\}$$

$$x_1 \in \{1, 2\}$$

$$x_2 \in \{1, 2, 3\}$$

$$x_3 \in \{3, 4\}$$

$$x_4 \in \{3, 4, 5\}$$

$$x_5 \in \{4, 5\}$$

$x_1 \in \{1, 2\}$

$x_2 \in \{1, 2, \cancel{3}\}$

Hall set

$x_3 \in \{3, 4\}$

$x_4 \in \{3, 4, 5\}$

$x_5 \in \{4, 5\}$

$x_1 \in \{1, 2\}$

$x_2 \in \{1, 2, \cancel{3}\}$

$x_3 \in \{3, 4\}$

Hall set $x_4 \in \{3, 4, 5\}$ **Explanation?**

$x_5 \in \{4, 5\}$

Lift explanation?

$$\langle x_3 \geq 3\rangle \wedge \langle x_3 \leq 4\rangle \wedge$$
$$\langle x_4 \geq 3\rangle \wedge \langle x_4 \leq 5\rangle \wedge$$
$$\langle x_5 \geq 4\rangle \wedge \langle x_3 \leq 5\rangle$$
$$\Rightarrow$$
$$\langle x_2 \neq 3\rangle$$

$x_1 \in \{1, 2\}$

$x_2 \in \{1, 2, \cancel{3}\}$

$x_3 \in \{3, 4\}$

$x_4 \in \{3, 4, 5\}$

$x_5 \in \{4, 5\}$

$$x_1 \in \{1, 2\}$$

$$x_2 \in \{1, 2, \cancel{3}\}$$

Lifted explanation

$$\langle x_3 \geq 3 \rangle \land \langle x_3 \leq 5 \rangle \;\land$$
$$\langle x_4 \geq 3 \rangle \land \langle x_4 \leq 5 \rangle \;\land$$
$$\langle x_5 \geq 3 \rangle \land \langle x_3 \leq 5 \rangle$$
$$\Rightarrow$$
$$\langle x_2 \neq 3 \rangle$$

$$x_3 \in \{3, 4\}$$

$$x_4 \in \{3, 4, 5\}$$

$$x_5 \in \{4, 5\}$$

$$
\begin{gathered}
\langle x_3 \geq 3\rangle \wedge \langle x_3 \leq 5\rangle \;\wedge \\
\langle x_4 \geq 3\rangle \wedge \langle x_4 \leq 5\rangle \;\wedge \\
\langle x_5 \geq 3\rangle \wedge \langle x_3 \leq 5\rangle \\
\Rightarrow \\
\langle x_2 \neq 3\rangle
\end{gathered}
$$

Convert into conflict explanation

$$\langle x_3 \geq 3 \rangle \land \langle x_3 \leq 5 \rangle \land$$
$$\langle x_4 \geq 3 \rangle \land \langle x_4 \leq 5 \rangle \land$$
$$\langle x_5 \geq 3 \rangle \land \langle x_3 \leq 5 \rangle \land$$
$$\langle x_3 = 3 \rangle$$
$$\Rightarrow$$
$$\perp$$

Reconstruct domains

$$\langle x_3 \geq 3 \rangle \wedge \langle x_3 \leq 5 \rangle \ \wedge$$
$$\langle x_4 \geq 3 \rangle \wedge \langle x_4 \leq 5 \rangle \ \wedge$$
$$\langle x_5 \geq 3 \rangle \wedge \langle x_3 \leq 5 \rangle \wedge$$
$$\langle x_3 = 3 \rangle$$
$$\Rightarrow$$
$$\perp$$

$$x_2 \in \{3\}$$

$$x_3 \in \{3, 4, 5\}$$

$$x_4 \in \{3, 4, 5\}$$

$$x_5 \in \{3, 4, 5\}$$

Reconstruct domains **Check condition**

$$\langle x_3 \geq 3\rangle \wedge \langle x_3 \leq 5\rangle \wedge \langle x_4 \geq 3\rangle \wedge \langle x_4 \leq 5\rangle \wedge \langle x_5 \geq 3\rangle \wedge \langle x_3 \leq 5\rangle \wedge \langle x_3 = 3\rangle \Rightarrow \bot$$

$$x_2 \in \{3\}$$

$$x_3 \in \{3, 4, 5\}$$

$$x_4 \in \{3, 4, 5\}$$

$$x_5 \in \{3, 4, 5\}$$

$$|\{3, 4, 5\}| < |\{x_2, x_3, x_4, x_5\}|$$

Explanation verified

Propagation for AllDifferent

Compute Hall sets

Prune based on Hall sets

Propagation for AllDifferent

Compute Hall sets

Prune based on Hall sets

All-Different Propagation

variables

x_1 1

x_2 2

x_3 3

x_4 4

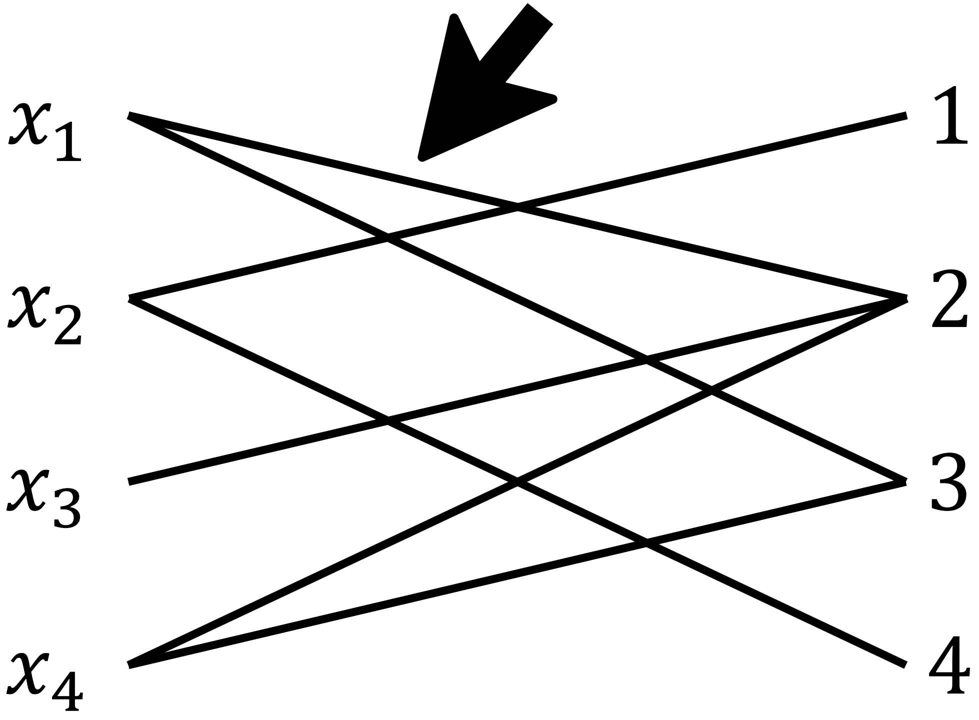
All-Different Propagation

Domain values

x_1	1
x_2	2
x_3	3
x_4	4

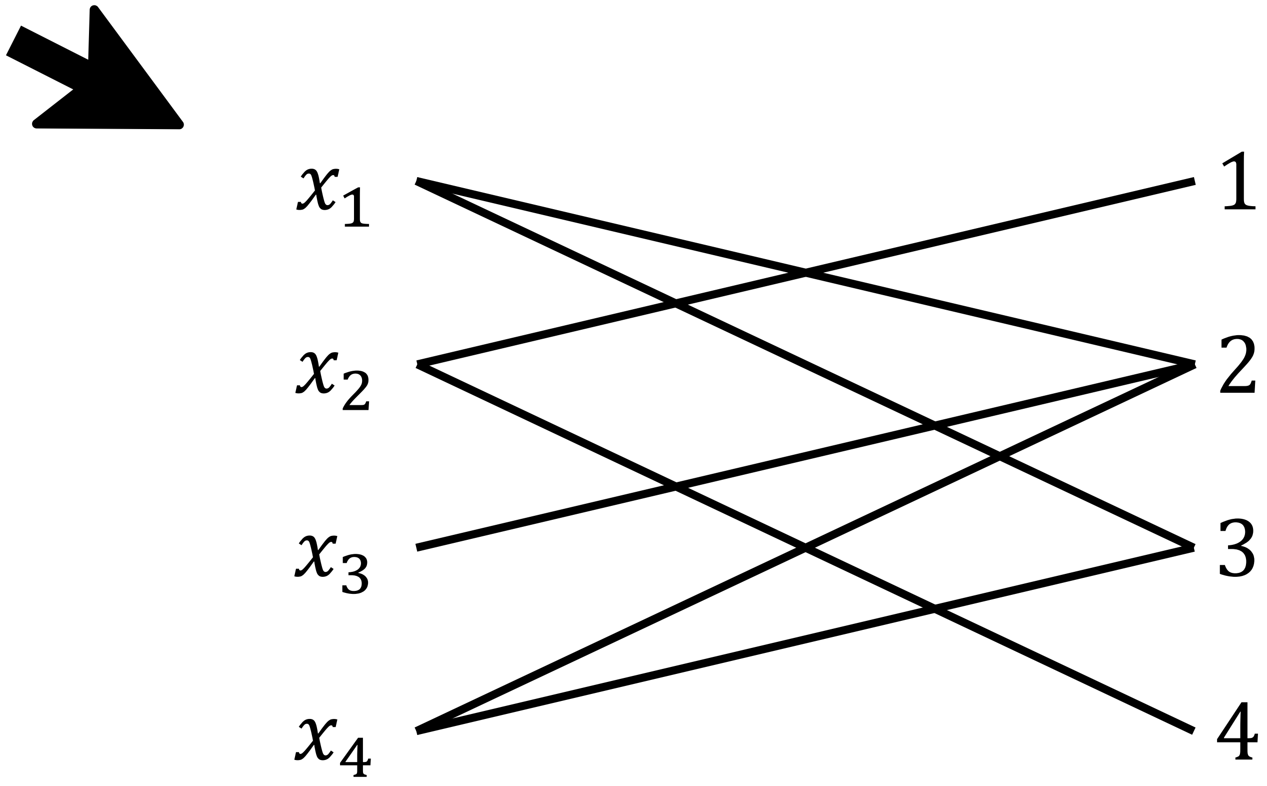
All-Different Propagation

edge connect variable to domain value



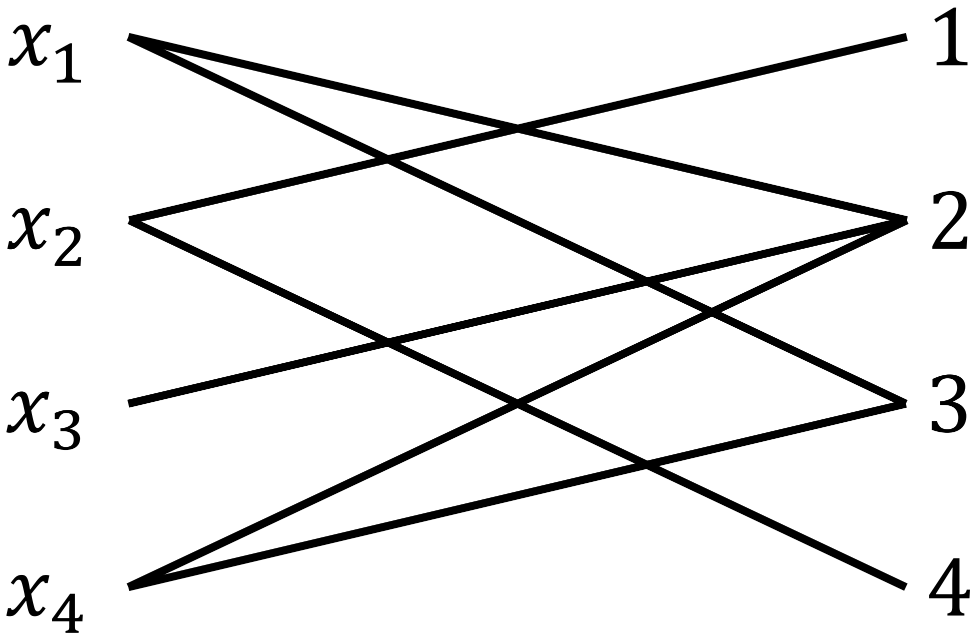
All-Different Propagation

Bi-partite graph



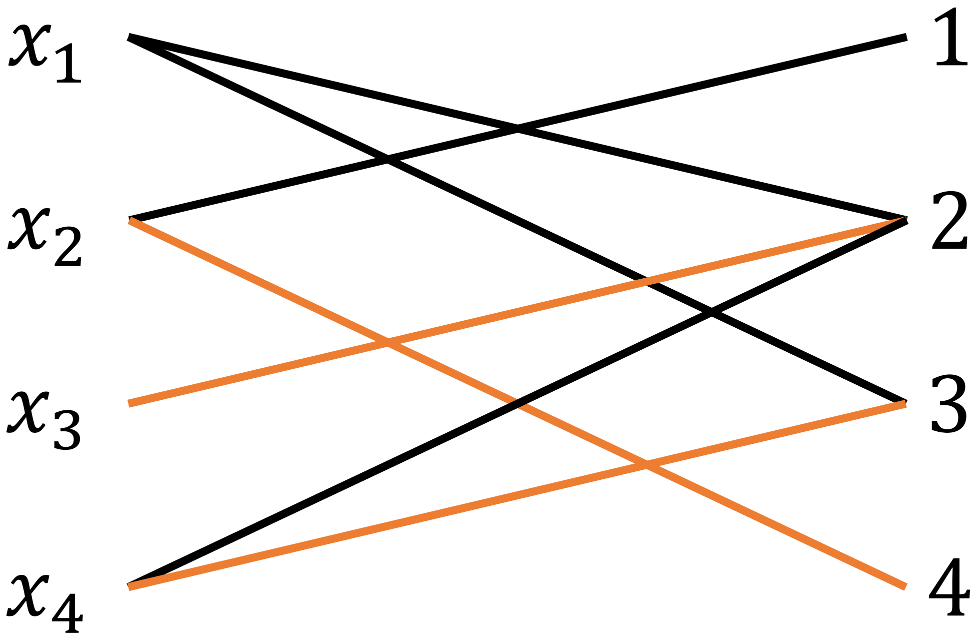
All-Different Propagation

“all-different is feasible if there exists a matching of size four”



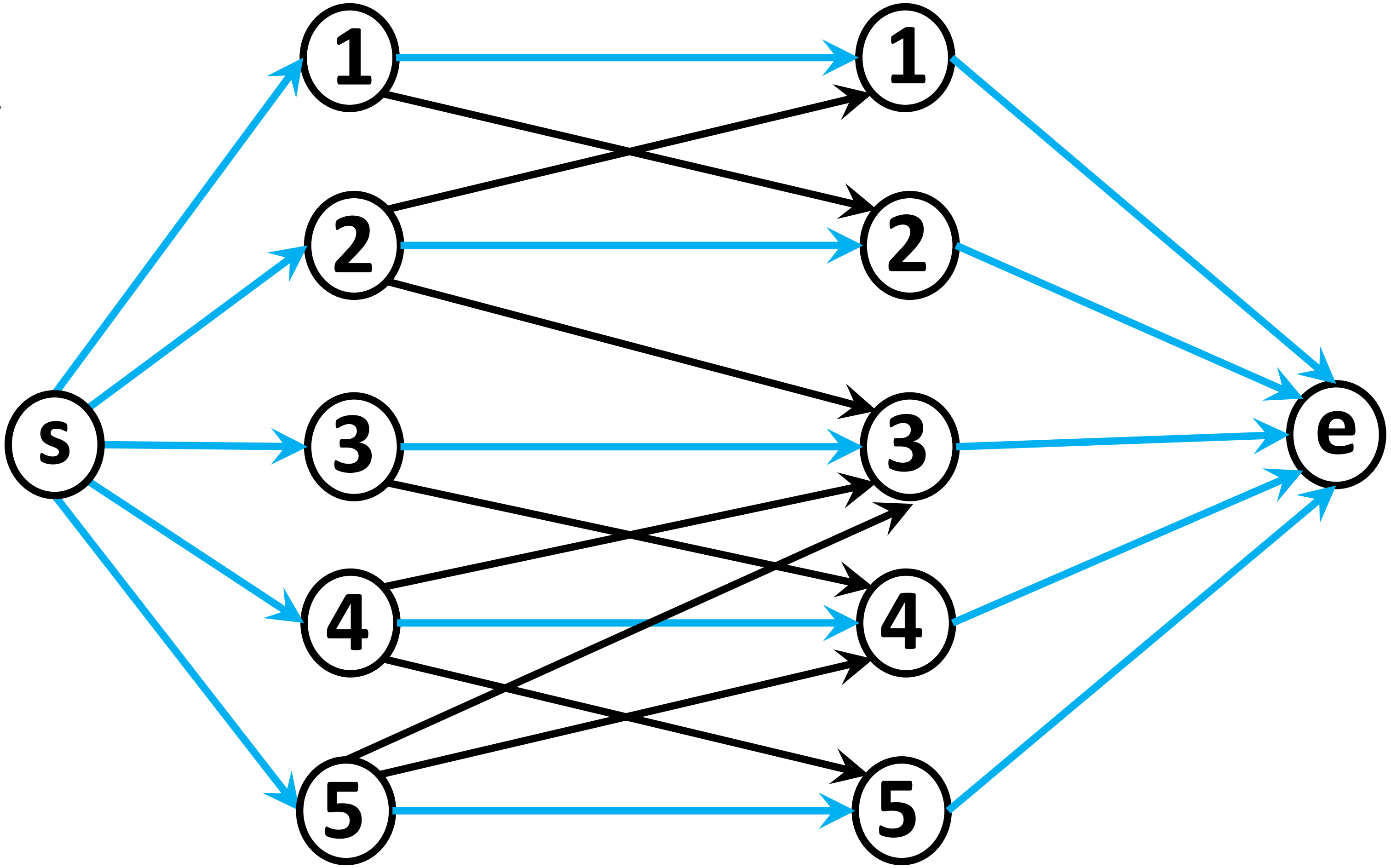
All-Different Propagation

Maximum matching is three, infeasible!



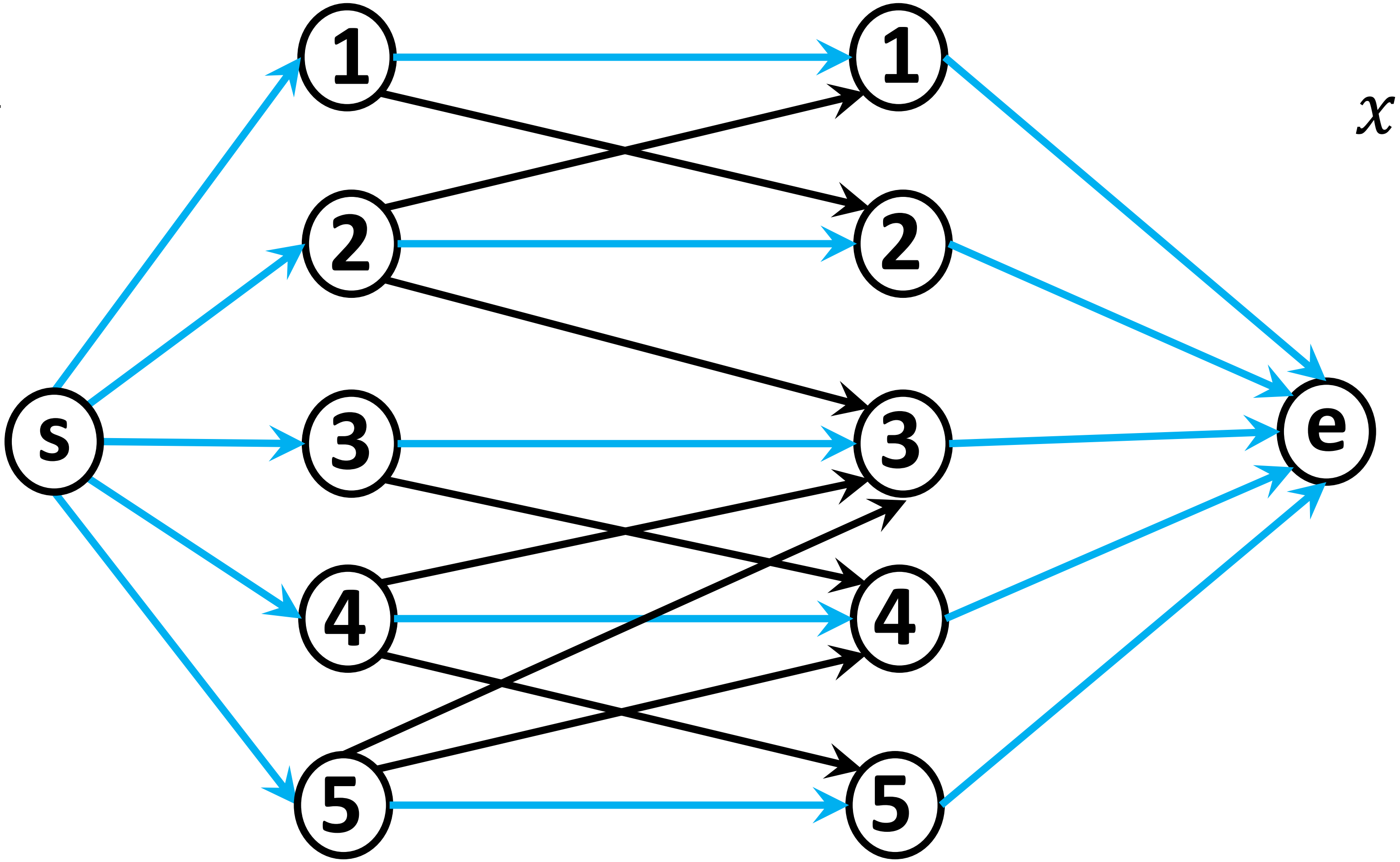
All-Different Flow Representation

$$x_1 \in \{1, 2\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{3, 4\}$$
$$x_4 \in \{3, 4, 5\}$$
$$x_5 \in \{4, 5\}$$



All-Different Flow Representation

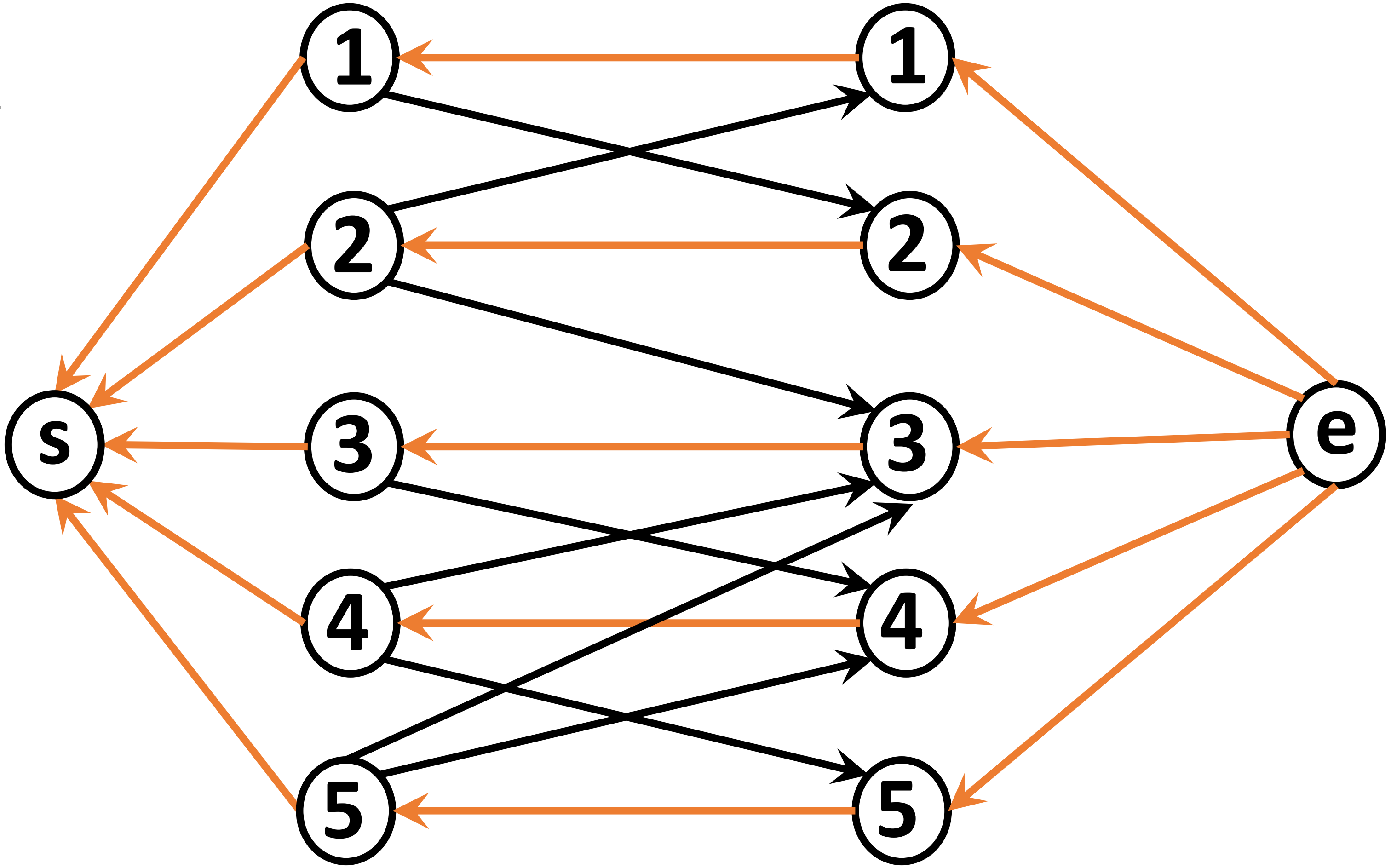
$x_1 \in \{1, 2\}$
$x_2 \in \{1, 2, 3\}$
$x_3 \in \{3, 4\}$
$x_4 \in \{3, 4, 5\}$
$x_5 \in \{4, 5\}$



$x_i = i$

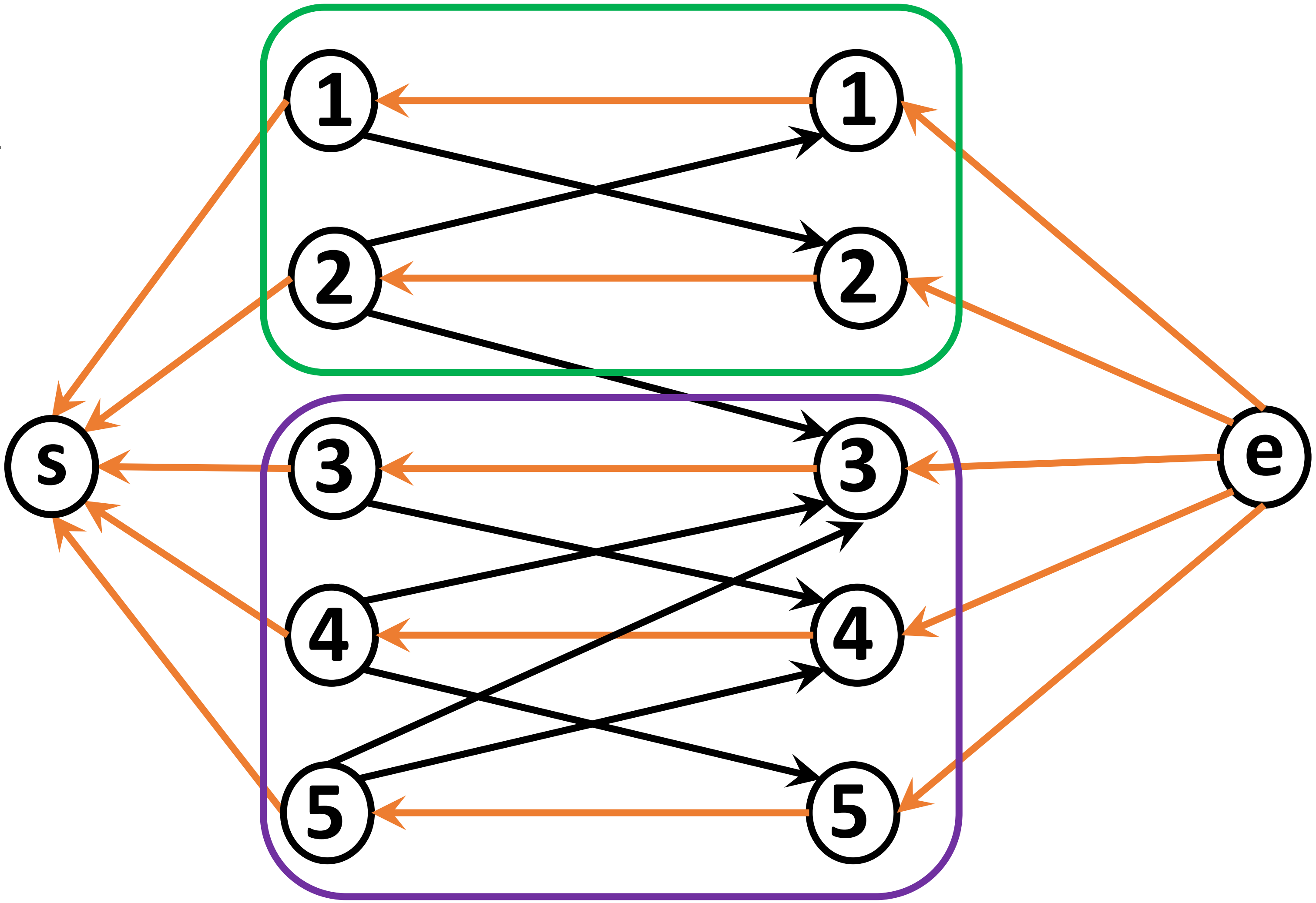
Residual graph: inverse flow edges

$x_1 \in \{1, 2\}$
$x_2 \in \{1, 2, 3\}$
$x_3 \in \{3, 4\}$
$x_4 \in \{3, 4, 5\}$
$x_5 \in \{4, 5\}$



Strongly Connected Components

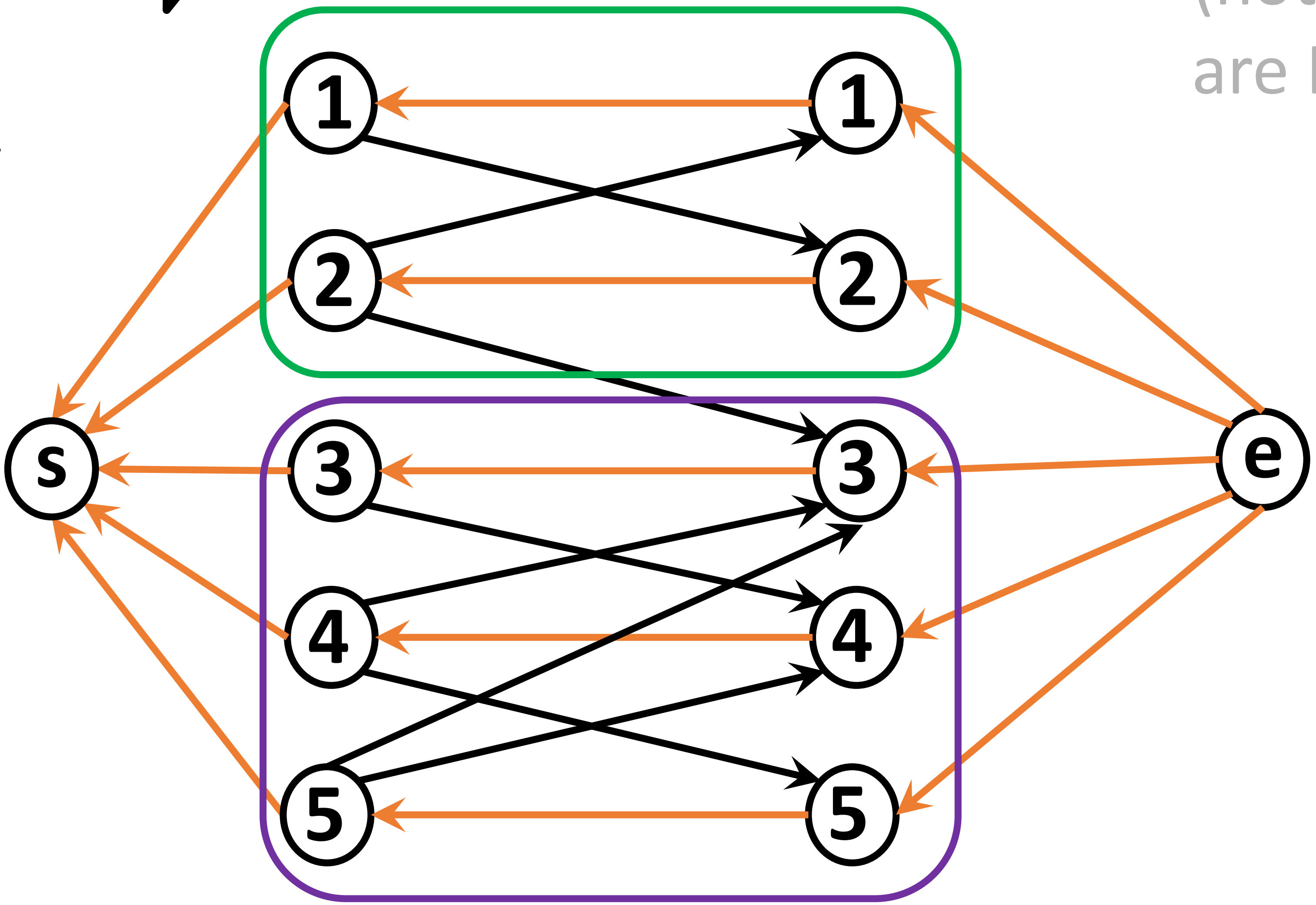
$$x_1 \in \{1, 2\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{3, 4\}$$
$$x_4 \in \{3, 4, 5\}$$
$$x_5 \in \{4, 5\}$$



$x_1 \in \{1, 2\}$
$x_2 \in \{1, 2, 3\}$
$x_3 \in \{3, 4\}$
$x_4 \in \{3, 4, 5\}$
$x_5 \in \{4, 5\}$

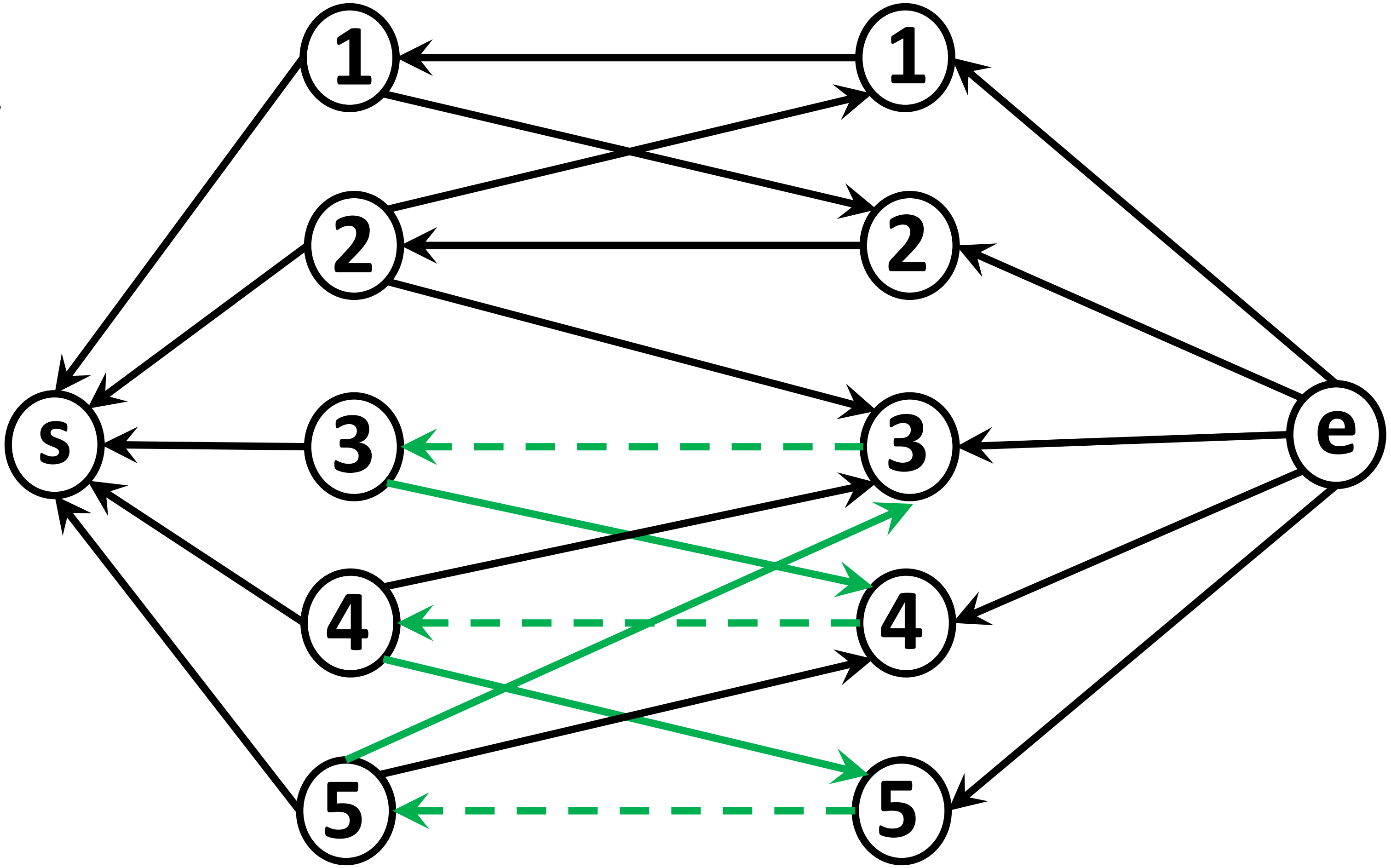
Hall sets form SCCs!

(not all SCCs are Hall sets)



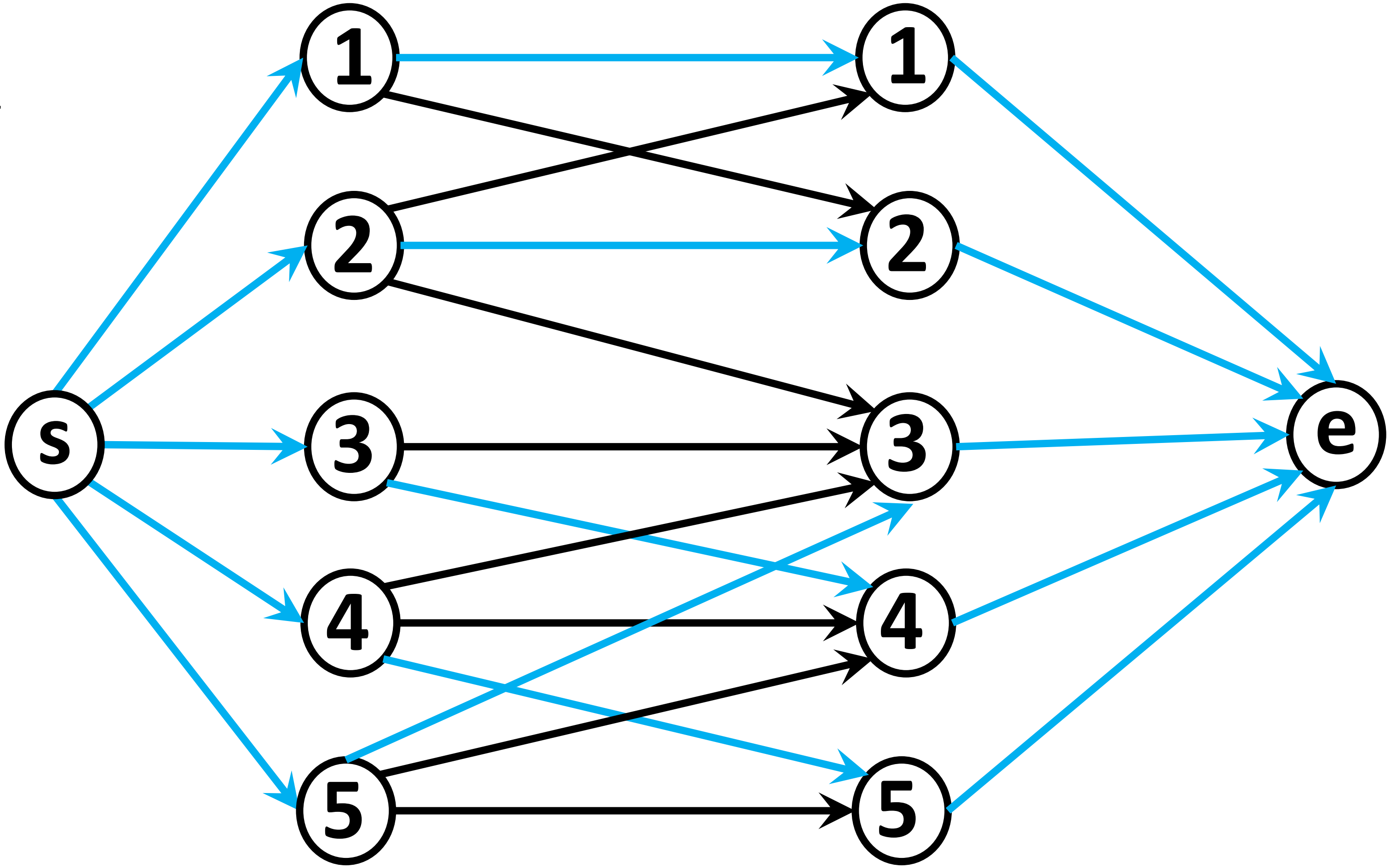
$$x_1 \in \{1, 2\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{3, 4\}$$
$$x_4 \in \{3, 4, 5\}$$
$$x_5 \in \{4, 5\}$$

Cycle within the SCC is a new solution!



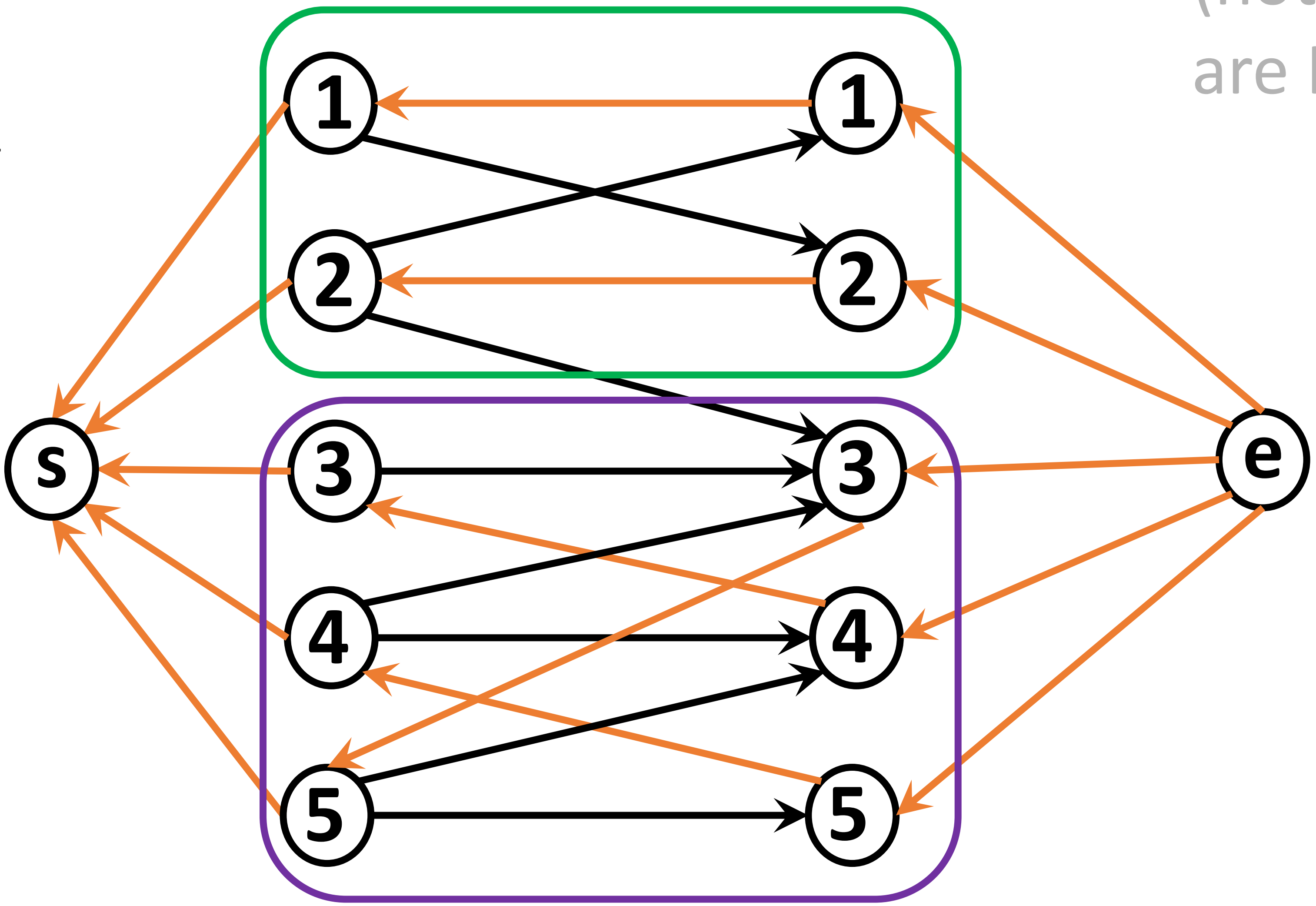
$$x_1 \in \{1, 2\}$$
$$x_2 \in \{1, 2, 3\}$$
$$x_3 \in \{3, 4\}$$
$$x_4 \in \{3, 4, 5\}$$
$$x_5 \in \{4, 5\}$$

Cycle within the SCC is a new solution!

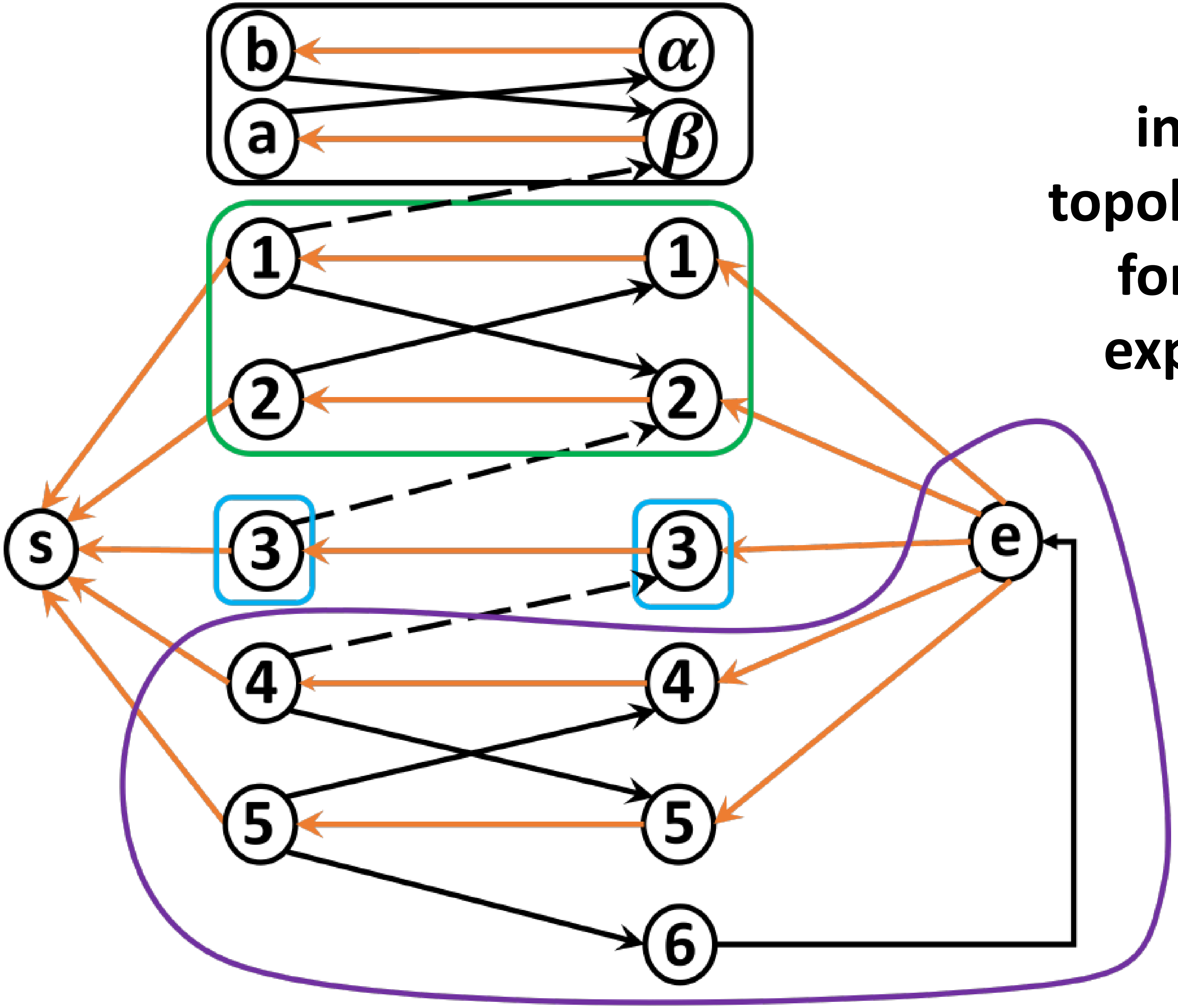


$x_1 \in \{1, 2\}$
$x_2 \in \{1, 2, 3\}$
$x_3 \in \{3, 4\}$
$x_4 \in \{3, 4, 5\}$
$x_5 \in \{4, 5\}$

Hall sets do not change!



(not all SCCs are Hall sets)



Prune in reverse topological order for smallest explanations

So far…

All-Different

Decomposition

Hall Sets

Checker → simple!

Propagation → Flow

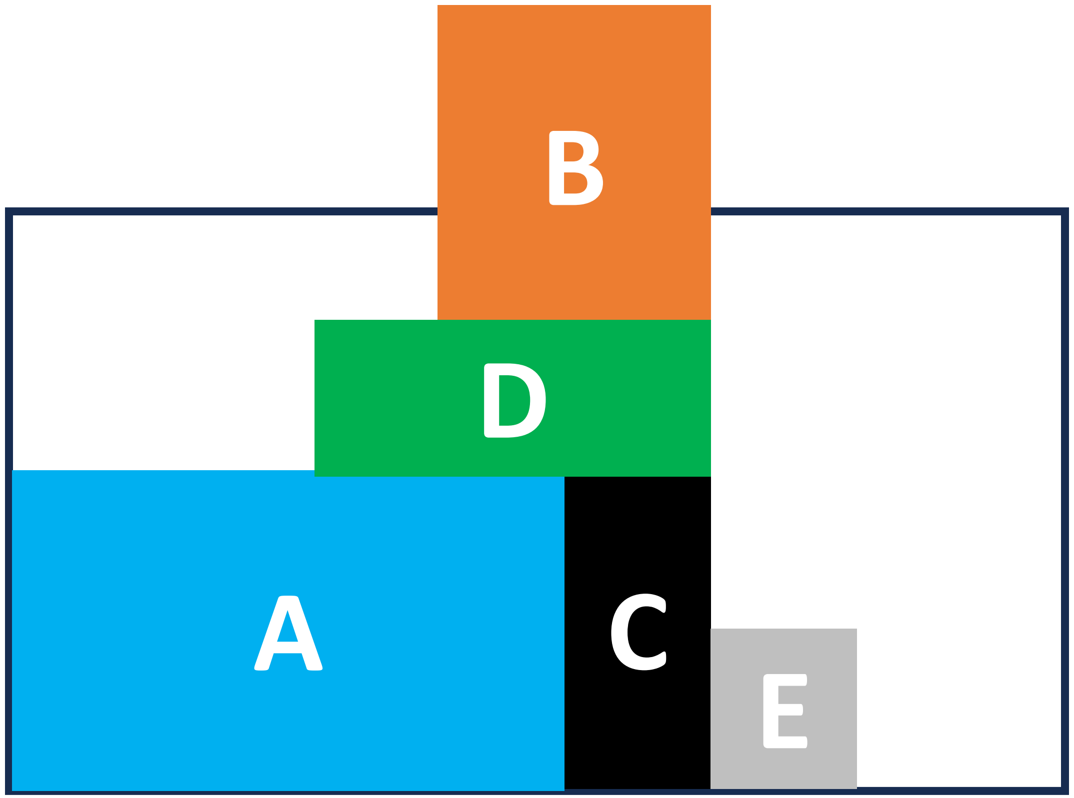
Up next…

Cumulative Constraint

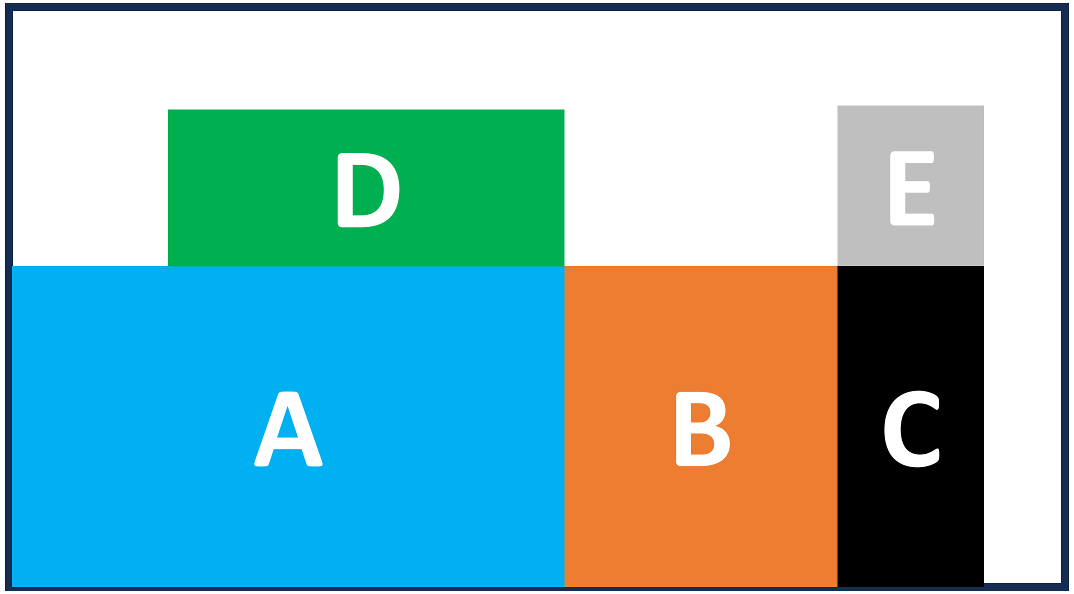
Timetable Reasoning

Energetic Reasoning

Can the smaller rectangles be placed within the large white rectangle?



Infeasible!



Feasible!

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

Integer variable representing possible starting times

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

Constant representing task duration

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

Constant representing resource requirements

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

Constant representing maximum number of available resources

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

“The resource consumption cannot exceed the maximum at any time”

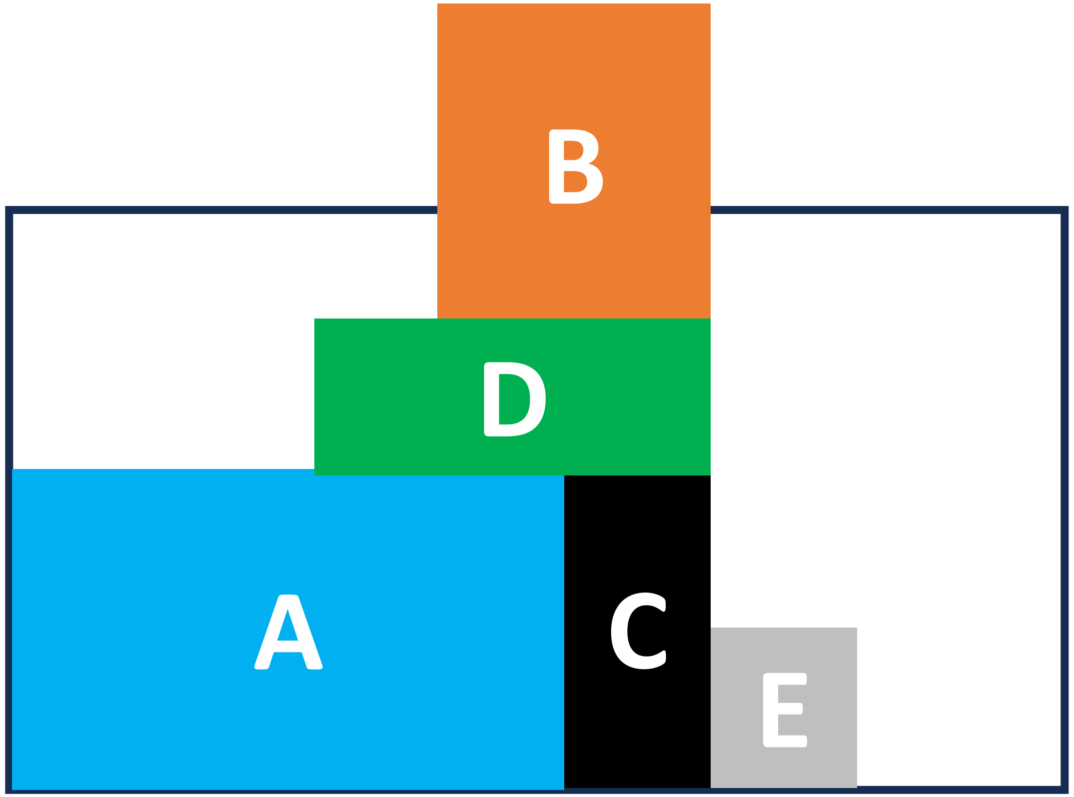
Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

“The resource consumption cannot exceed the maximum at any time”

Infeasible!



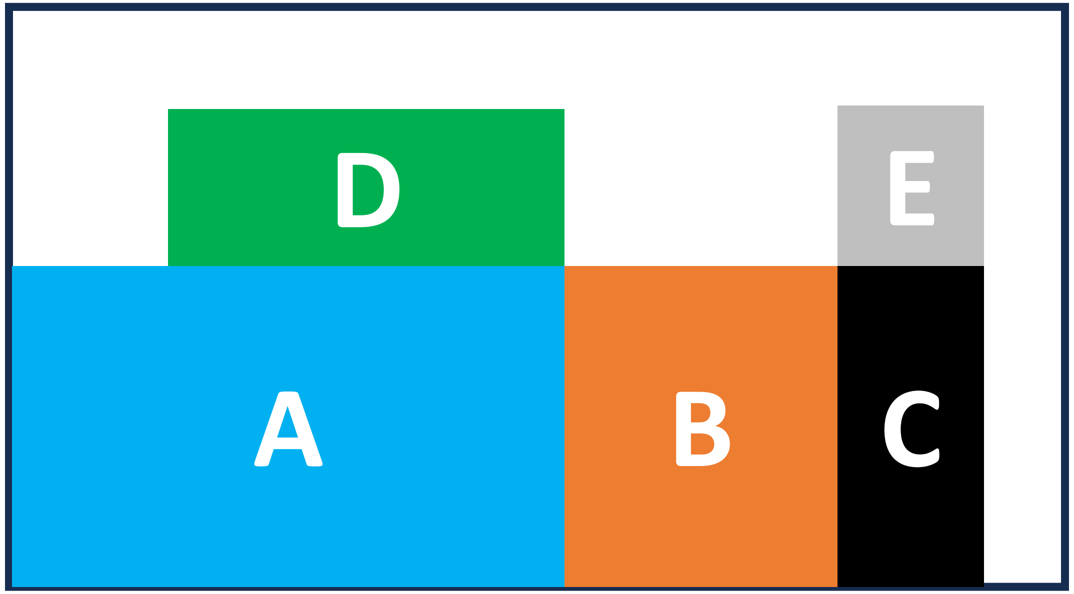
Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

“The resource consumption cannot exceed the maximum at any time”

Feasible!



Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

"The resource consumption cannot exceed the maximum at any time"

Determining the existence of a feasible solution for cumulative is NP-complete

(Cumulative appears as a subproblem of other NP-hard problems!)

Cumulative Constraint

$$task_i = (s_i, D_i, R_i)$$

$$R_{max}$$

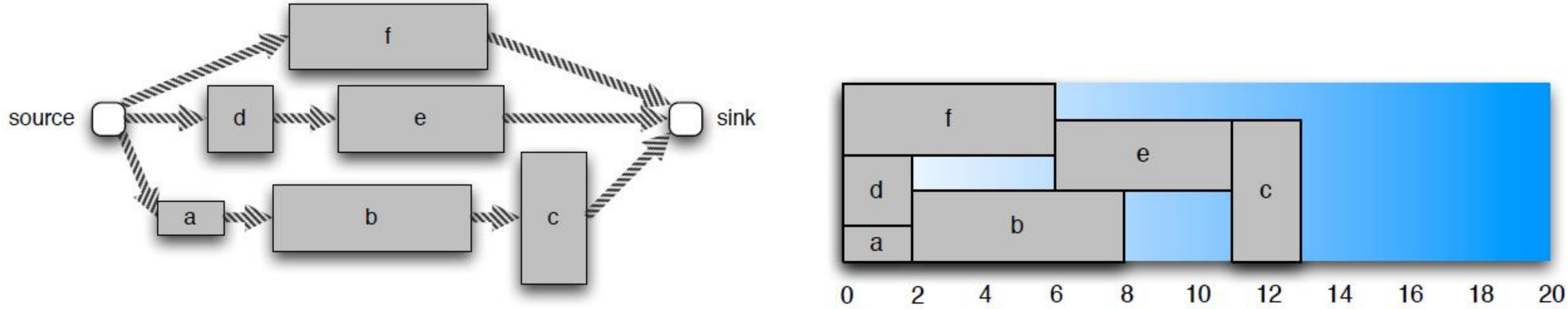
"The resource consumption cannot exceed the maximum at any time"

Determining the existence of a feasible solution for cumulative is NP-complete

…but since it appears as a common subproblem,
let us try to do some propagation in polynomial time

Combine multiple propagators to solve complex problems!

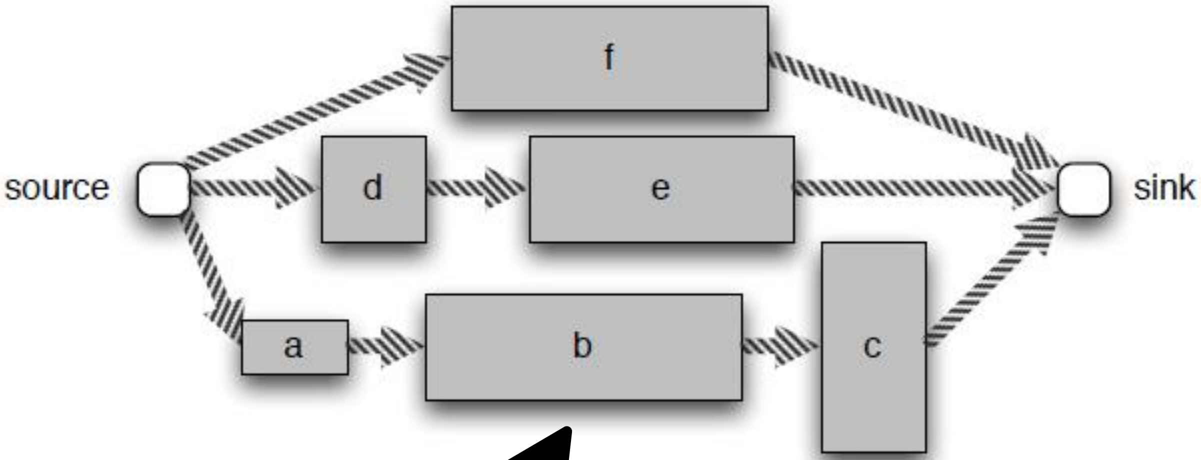
Resource-Constrained Project Scheduling Problem

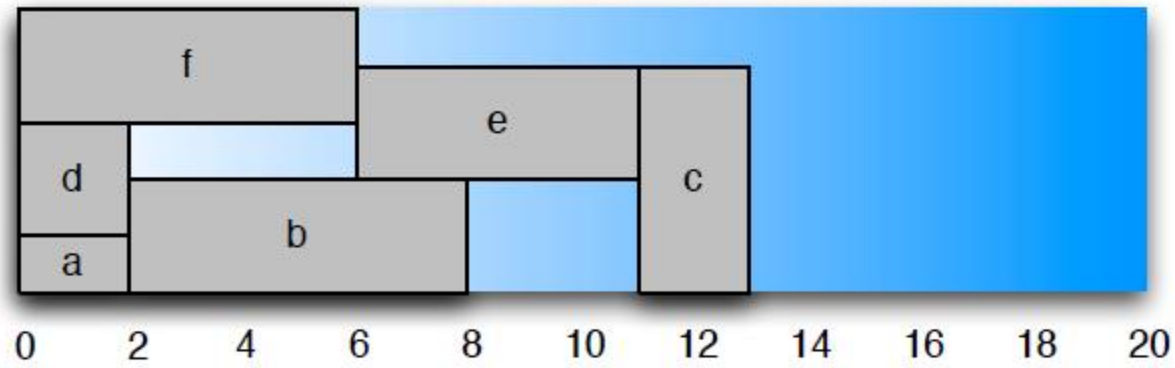


Picture taken from the PhD thesis of Andreas Schutt (2011)

Combine multiple propagators to solve complex problems!

Resource-Constrained Project Scheduling Problem





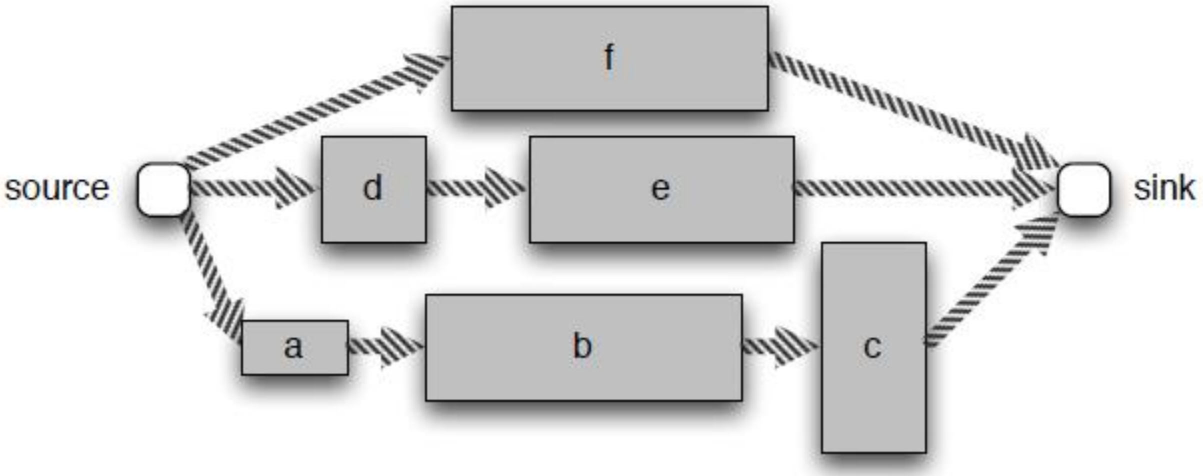
Linear inequality propagators
for precedence constraints

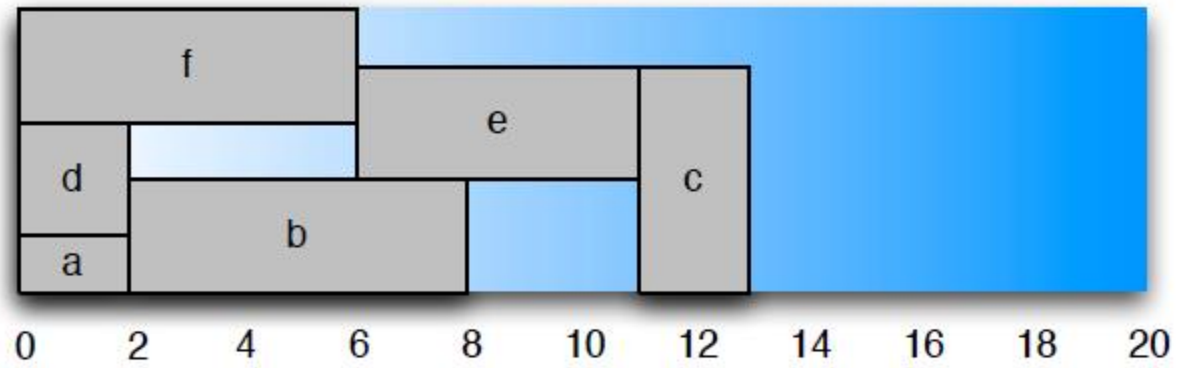
$$s_i - s_j \geq D_i$$

Picture taken from the PhD thesis of Andreas Schutt (2011)

Combine multiple propagators to solve complex problems!

Resource-Constrained Project Scheduling Problem





(Multiple) cumulative constraints for resource constraints

Picture taken from the PhD thesis of Andreas Schutt (2011)

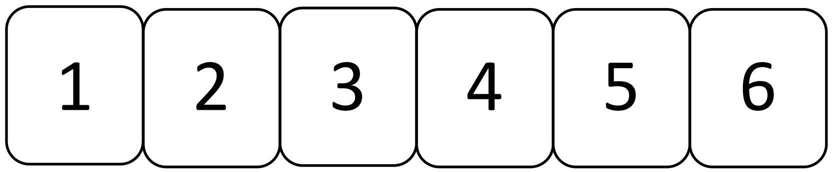
Variant 1:
Timetabling reasoning for Cumulative

Cumulative Propagator

$$s = \{1, 2, 3\}$$

$$D = 4$$

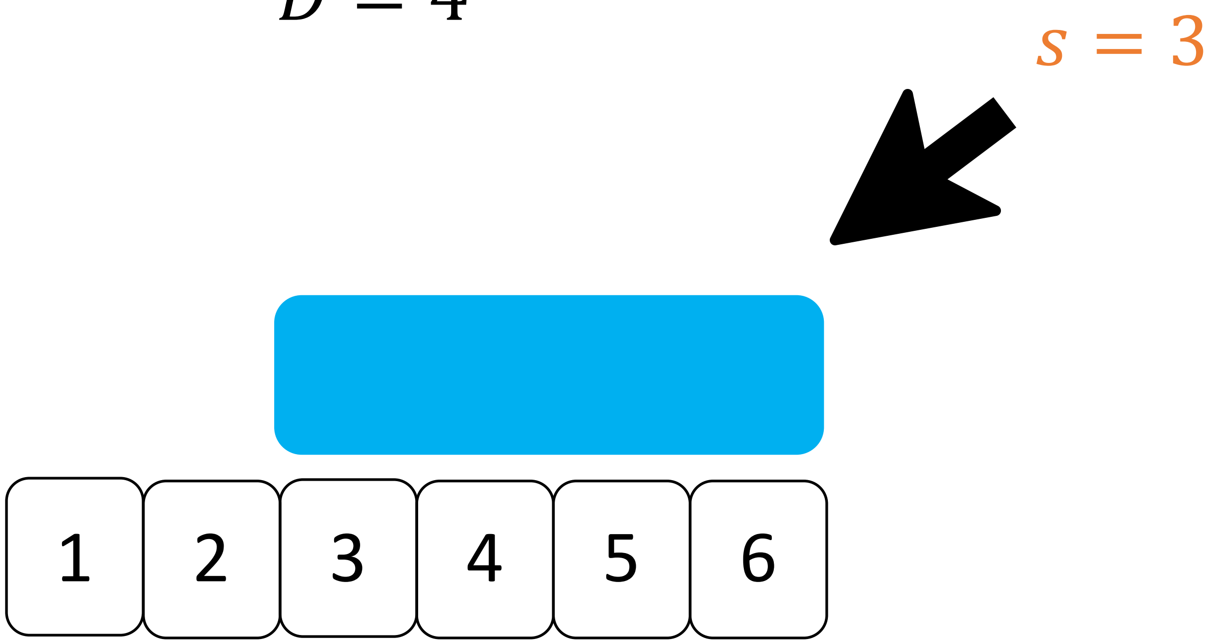
Integer variable representing the start time of the task



Cumulative Propagator

$$s = \{1, 2, 3\}$$

$$D = 4$$

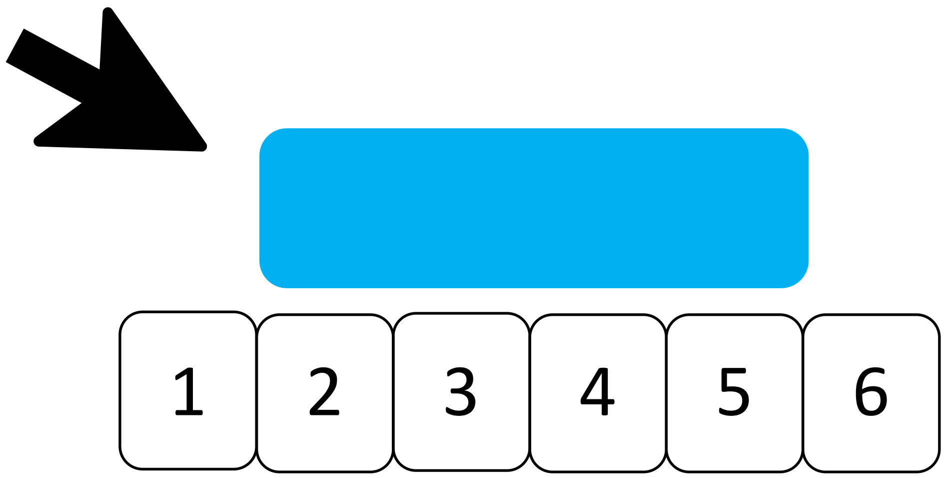


Cumulative Propagator

$$s = \{1, 2, 3\}$$

$$D = 4$$

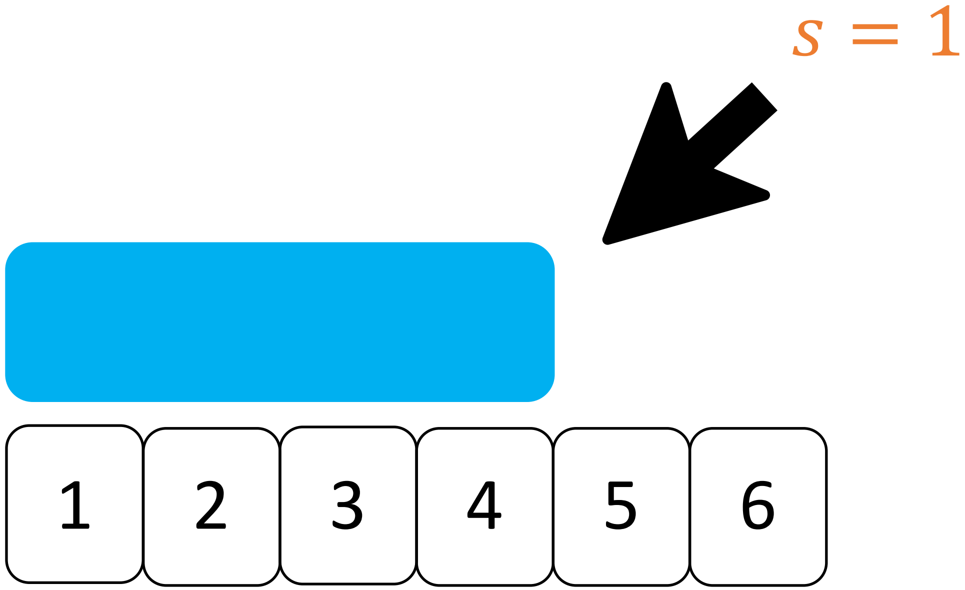
$s = 2$



Cumulative Propagator

$$s = \{1, 2, 3\}$$

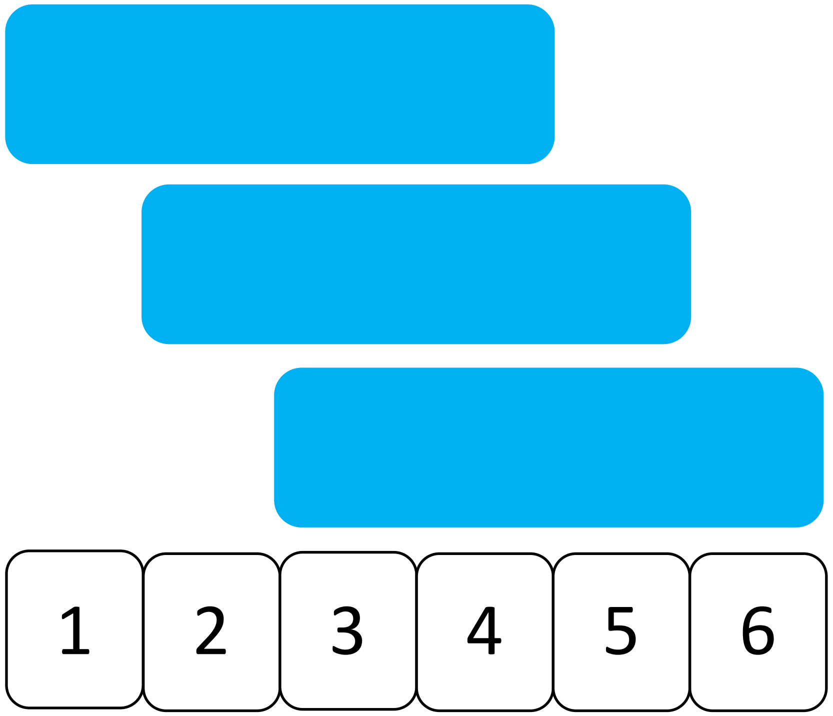
$$D = 4$$



Cumulative Propagator

$$s = \{1, 2, 3\}$$

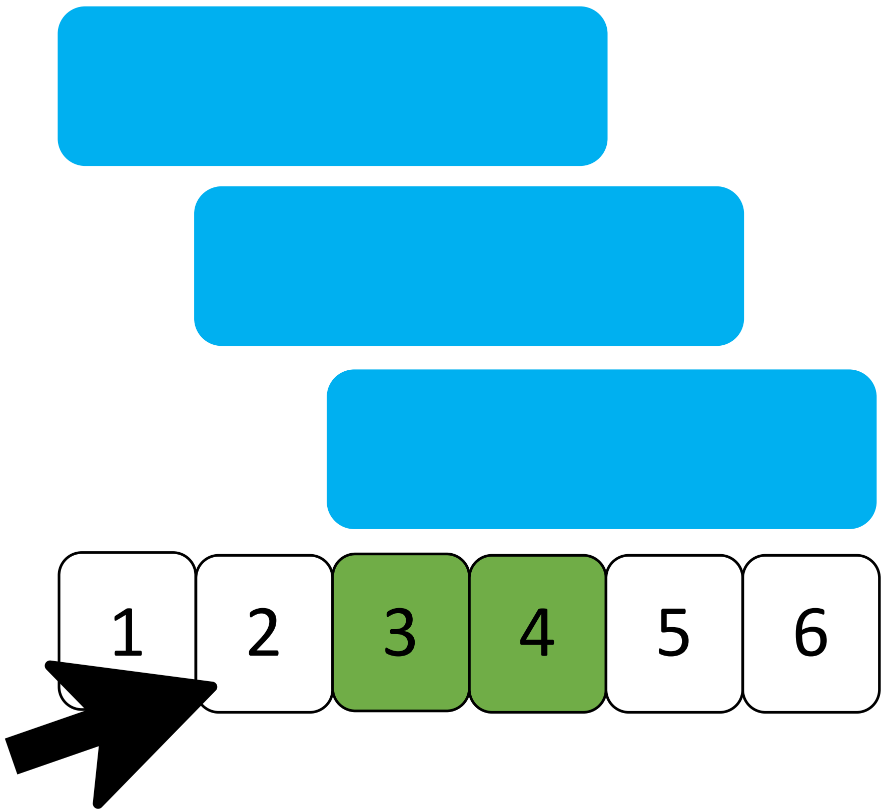
$$D = 4$$



Cumulative Propagator

$$s = \{1, 2, 3\}$$

$$D = 4$$



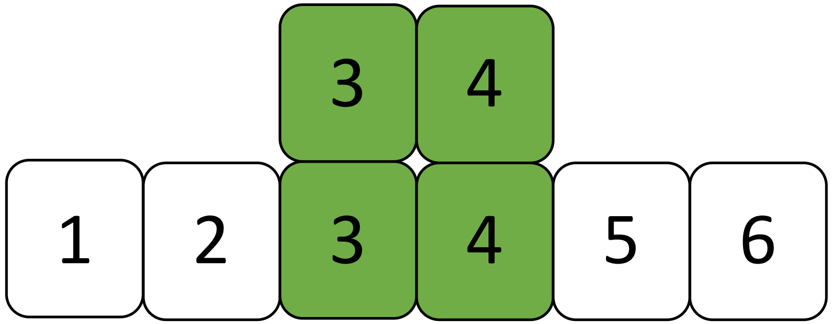
Compulsory
resource consumption

Cumulative Propagator

$$s_1, s_2 = \{1, 2, 3\}$$

$$D_1, D_2 = 4$$

Two tasks

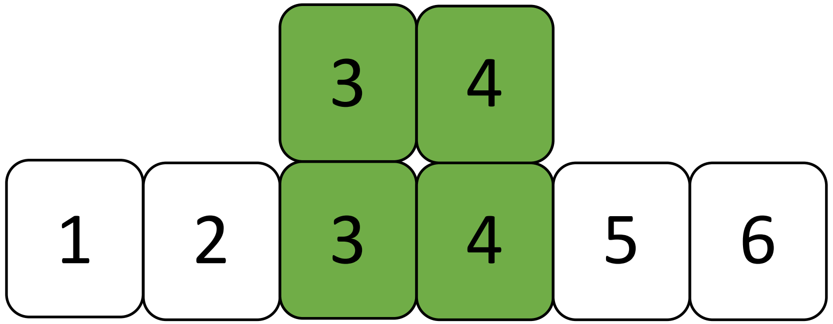


Cumulative Propagator

$$s_1, s_2 = \{1, 2, 3\}$$

$$D_1, D_2 = 4$$

$$R_{max} = 1$$



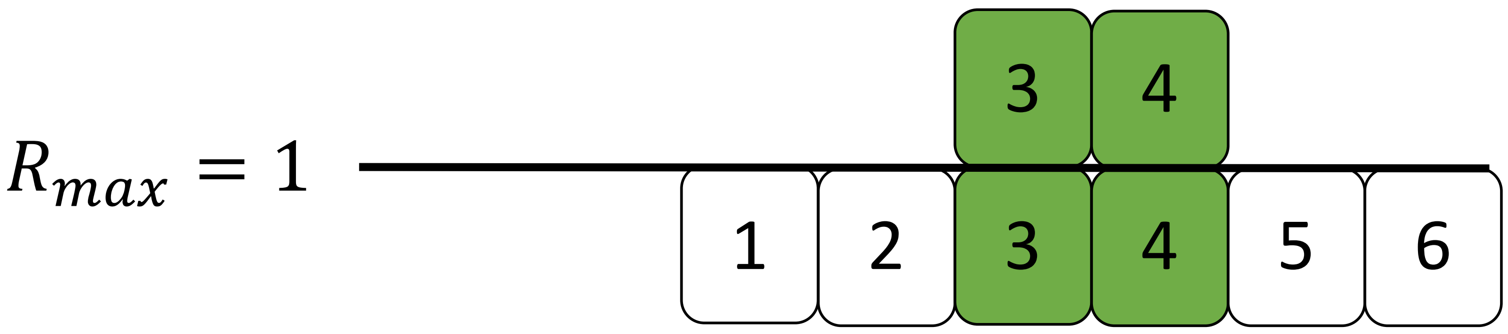
Cumulative Propagator

$$s_1, s_2 = \{1, 2, 3\}$$

$$D_1, D_2 = 4$$

$$R_{max} = 1$$

Compulsory part reasoning reveals infeasibility



Cumulative Propagator

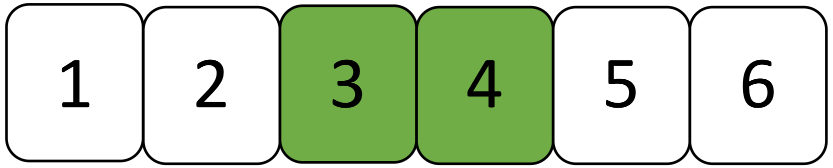
$$s_1, s_2 = \{1, 2, 3\}$$

$$D_1, D_2 = 4$$

$$R_{max} = 1$$

$$f(\{1, 2, 3\}, \{1, 2, 3\}) = (\{\quad\}, \{\quad\})$$

empty domains,
no feasible solution
exists

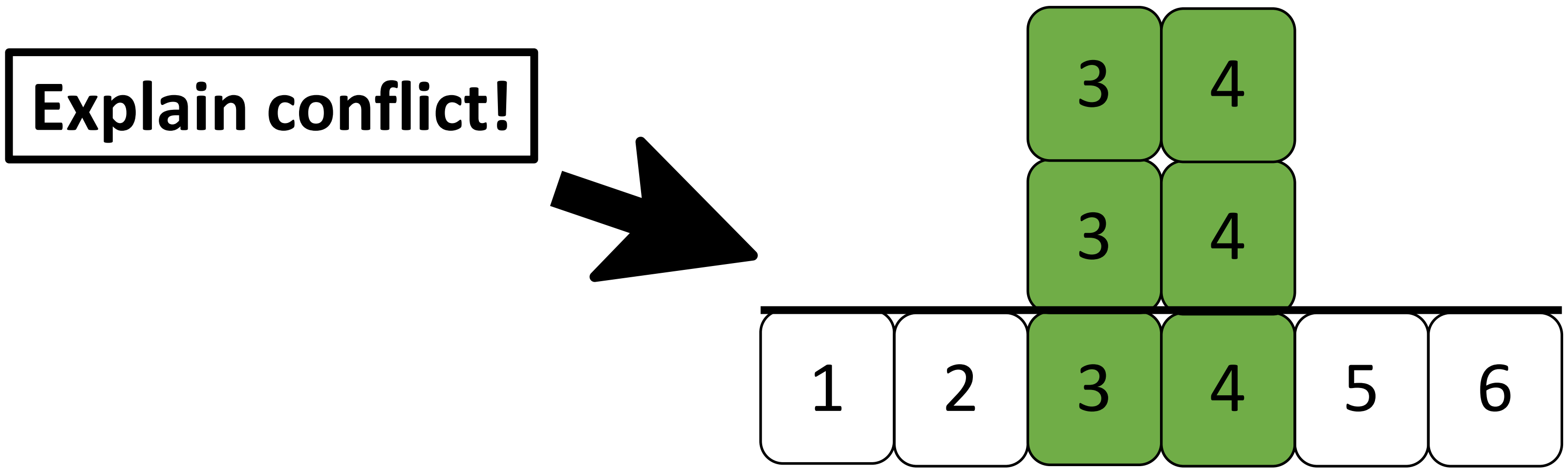


Explaining Timetabling

$$s_1, s_2, s_3 = \{1, 2, 3\}$$

$$D_1, D_2, D_3 = 4$$

$$R_{max} = 1$$



Explaining Timetabling

$$s_1, s_2, s_3 = \{1, 2, 3\}$$

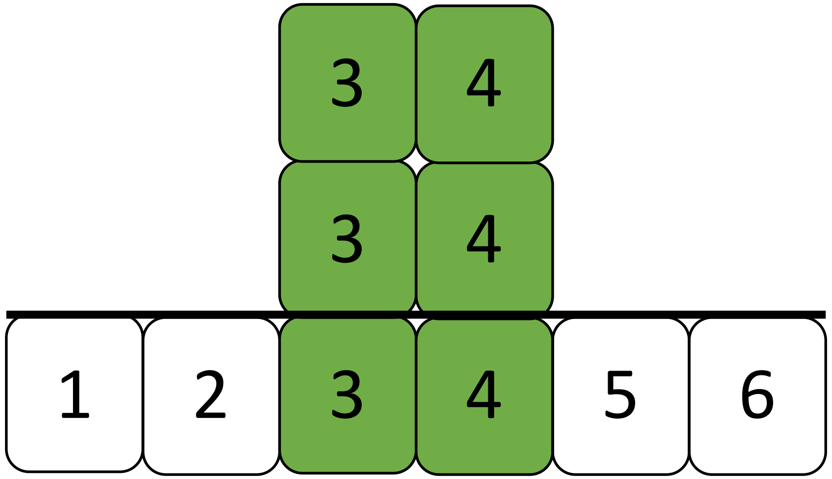
$$D_1, D_2, D_3 = 4$$

$$R_{max} = 1$$

$$[s_i \geq 1] \wedge [s_i \leq 3] \wedge [s_j \geq 1] \wedge [s_j \leq 3] \rightarrow \perp$$

$$(i, j) \in \{(1,2), (1,3), (2,3)\}$$

Three possible explanations!



Explaining Timetabling

Lifting?

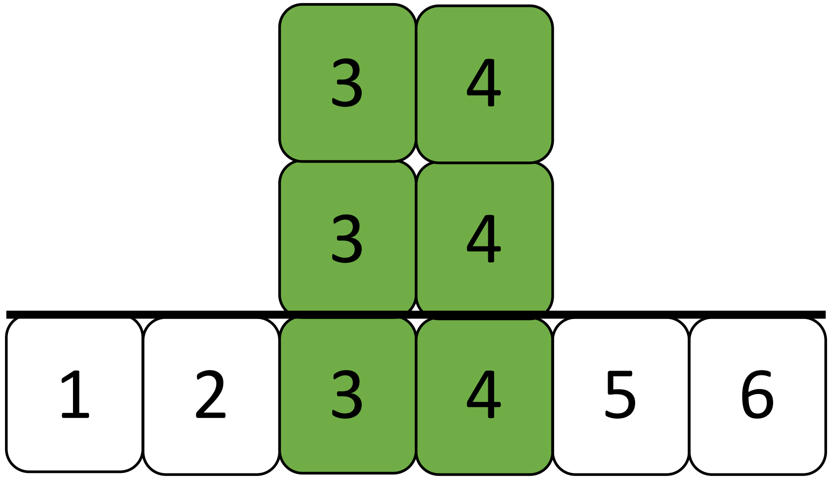
$$s_1, s_2, s_3 = \{1, 2, 3\}$$

$$D_1, D_2, D_3 = 4$$

$$R_{max} = 1$$

$$[s_i \geq 1] \wedge [s_i \leq 3] \wedge [s_j \geq 1] \wedge [s_j \leq 3] \rightarrow \perp$$

$$(i, j) \in \{(1,2), (1,3), (2,3)\}$$



Explaining Timetabling

Lifted explanation

$$s_1, s_2, s_3 = \{1, 2, 3\}$$

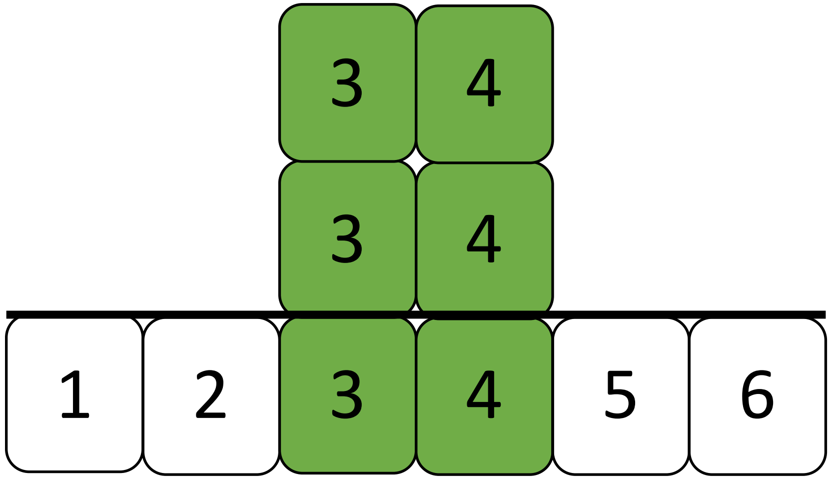
$$D_1, D_2, D_3 = 4$$

$$R_{max} = 1$$

$[s_i \geq 0]$ $[s_j \geq 0]$

$$\cancel{[s_i \geq 1]} \wedge [s_i \leq 3] \wedge \cancel{[s_j \geq 1]} \wedge [s_j \leq 3] \rightarrow \perp$$

$$(i, j) \in \{(1,2), (1,3), (2,3)\}$$

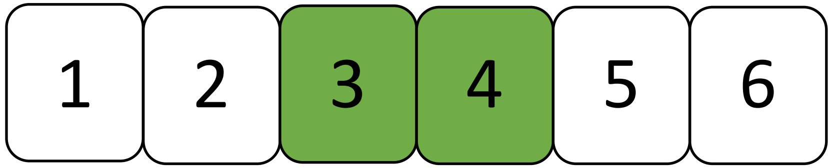


Propagation

$$s_1 = \{1, 2, 3\}, s_2 = \{2, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$

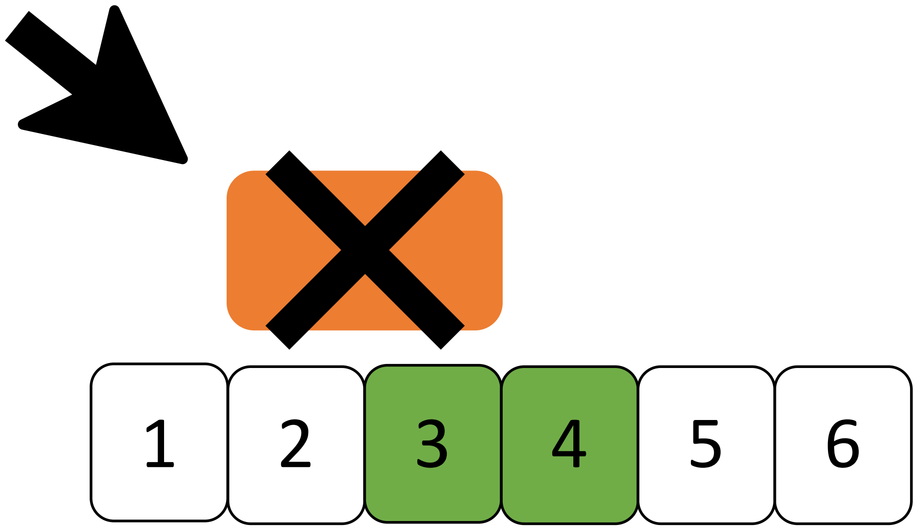


Propagation

$$s_1 = \{1,2,3\}, s_2 = \{2,3,4,5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$



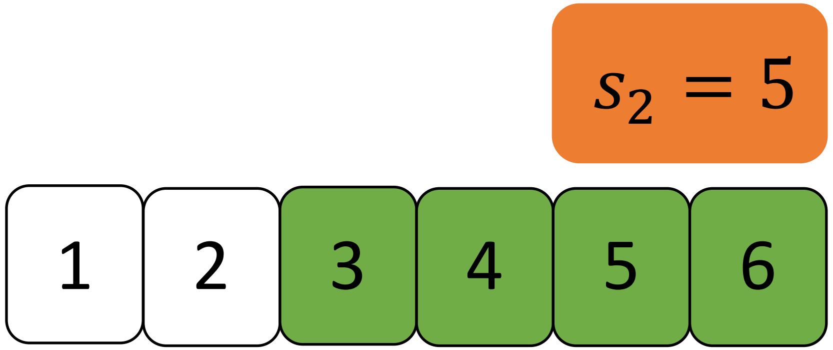
Propagation

$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2, 3, 4,} 5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$

Compulsory part reasoning forces task 2 to start at time 5



Propagation

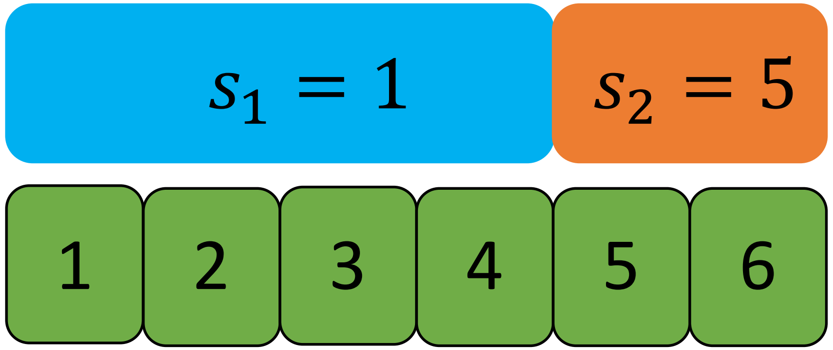
$$s_1 = \{1, \cancel{2, 3}\}, s_2 = \{\cancel{2, 3, 4,} 5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$

Compulsory part reasoning reveals the only feasible assignment

“timetable reasoning”



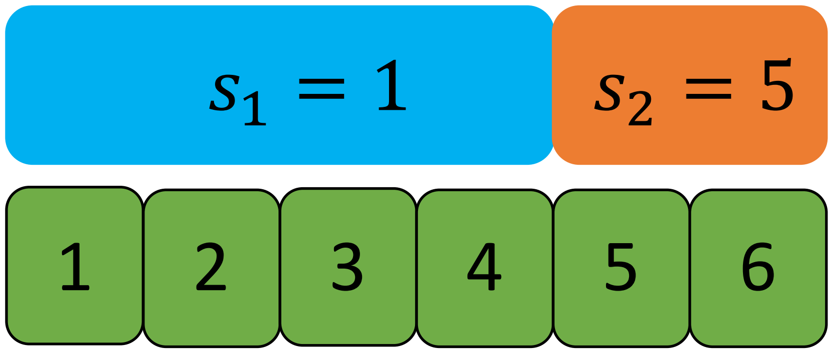
Propagation

$$s_1 = \{1,2,3\}, s_2 = \{2,3,4,5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$

$$f(\{1,2,3\},\{2,3,4,5\}) = (\{1\},\{5\})$$

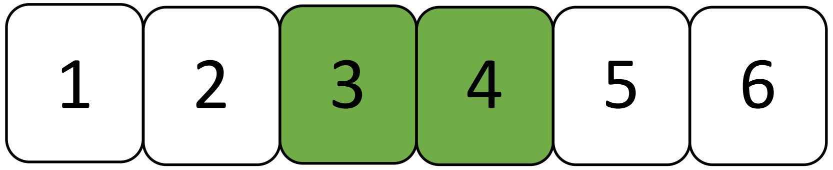


Explaining Timetabling

$$s_1 = \{1, 2, 3\}, s_2 = \{2, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$$R_{max} = 1$$



Explaining Timetabling

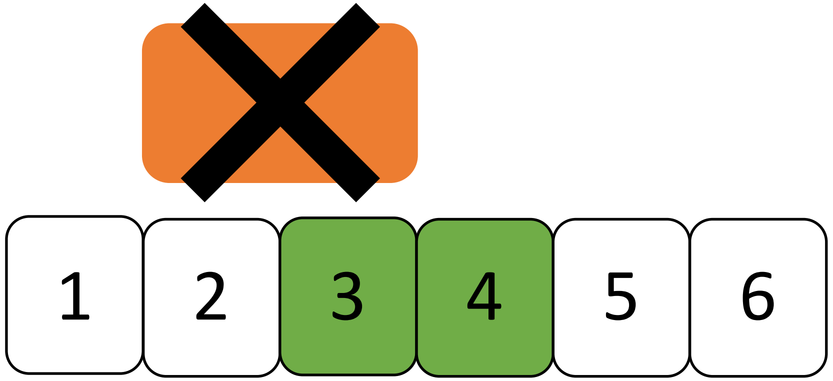
$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2}, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$s_2 \neq 2$

$$R_{max} = 1$$

$\boxed{\text{Explain } s_2 \neq 2}$

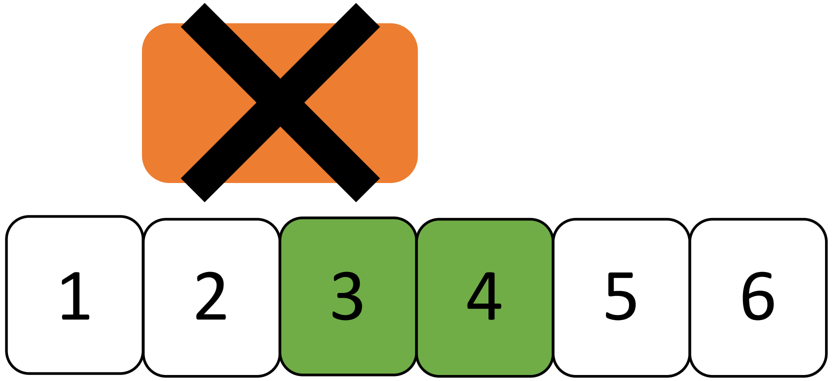


Explaining Timetabling

$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2}, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$$s_2 \neq 2 \qquad R_{max} = 1$$



$\boxed{\textbf{Explain } s_2 \neq 2}$

$$[s_1 \geq 1] \wedge [s_1 \leq 3] \rightarrow [s_2 \neq 2]$$

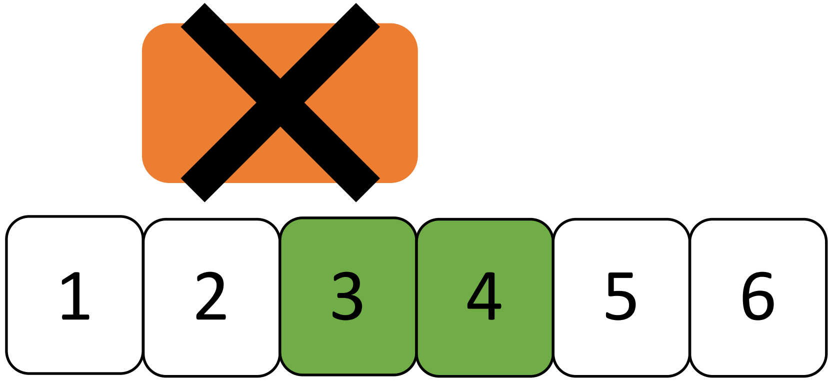
Explaining Timetabling

$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2}, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$s_2 \neq 2$

$$R_{max} = 1$$



Lifting?

$$[s_1 \geq 1] \wedge [s_1 \leq 3] \rightarrow [s_2 \neq 2]$$

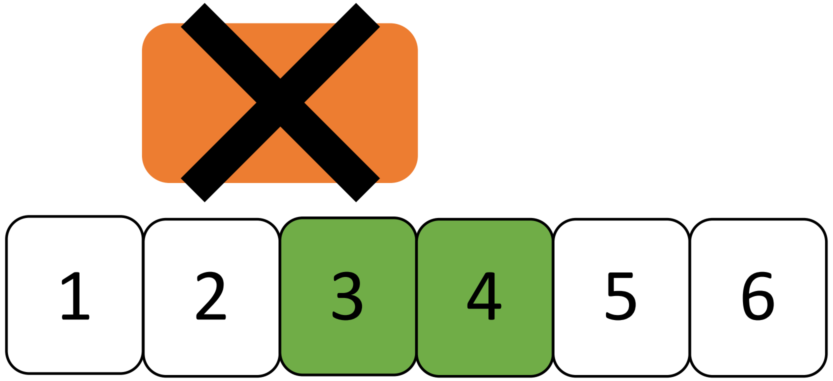
Explaining Timetabling

$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2}, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$s_2 \neq 2$

$$R_{max} = 1$$



(The lifted explanation only makes sense if $'s_1 \geq 0'$ is reasonable for this problem)

Lifted explanation!

$$[s_1 \geq 1] \wedge [s_1 \leq 3] \rightarrow [s_2 \neq 2]$$

$$[s_1 \geq 0] \wedge [s_1 \leq 3] \rightarrow [s_2 \neq 2]$$

Explaining Timetabling

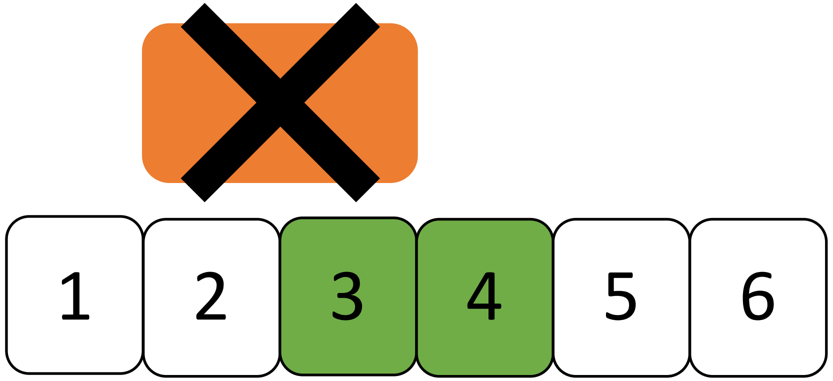
$$s_1 = \{1, 2, 3\}, s_2 = \{\cancel{2}, 3, 4, 5\}$$

$$D_1 = 4, D_2 = 2$$

$s_2 \neq 2$

$$R_{max} = 1$$

In practice, scheduling uses interval variables

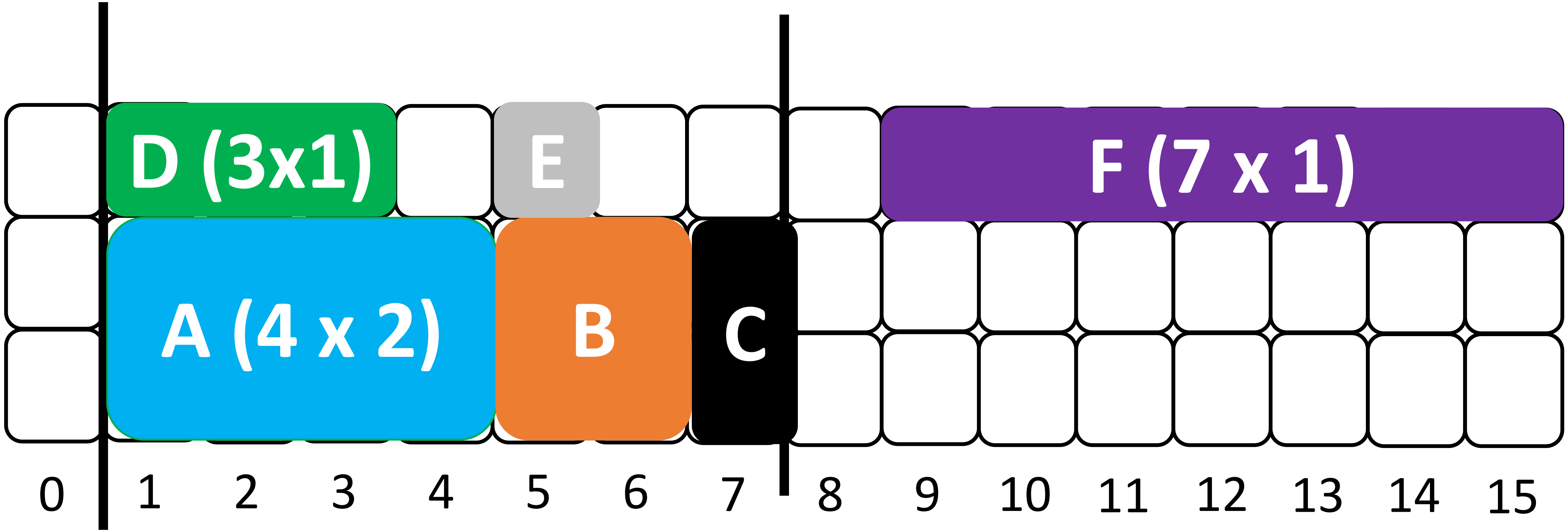


$$[s_1 \geq 0] \land [s_1 \leq 3] \land [s_2 \geq 2] \rightarrow [s_2 \geq 5]$$

(only bounds in explanations)

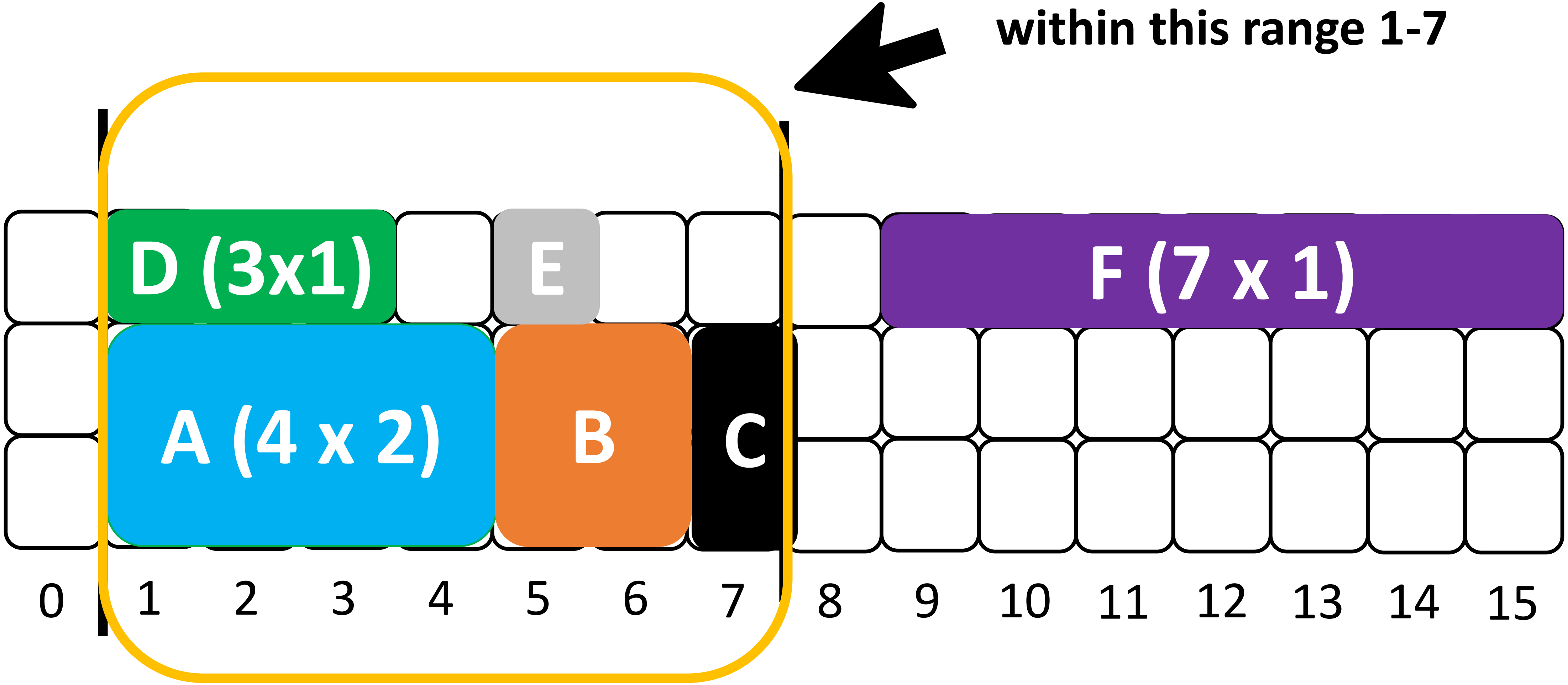
Variant 2:
Energetic Reasoning
for Cumulative

Three units of resource available



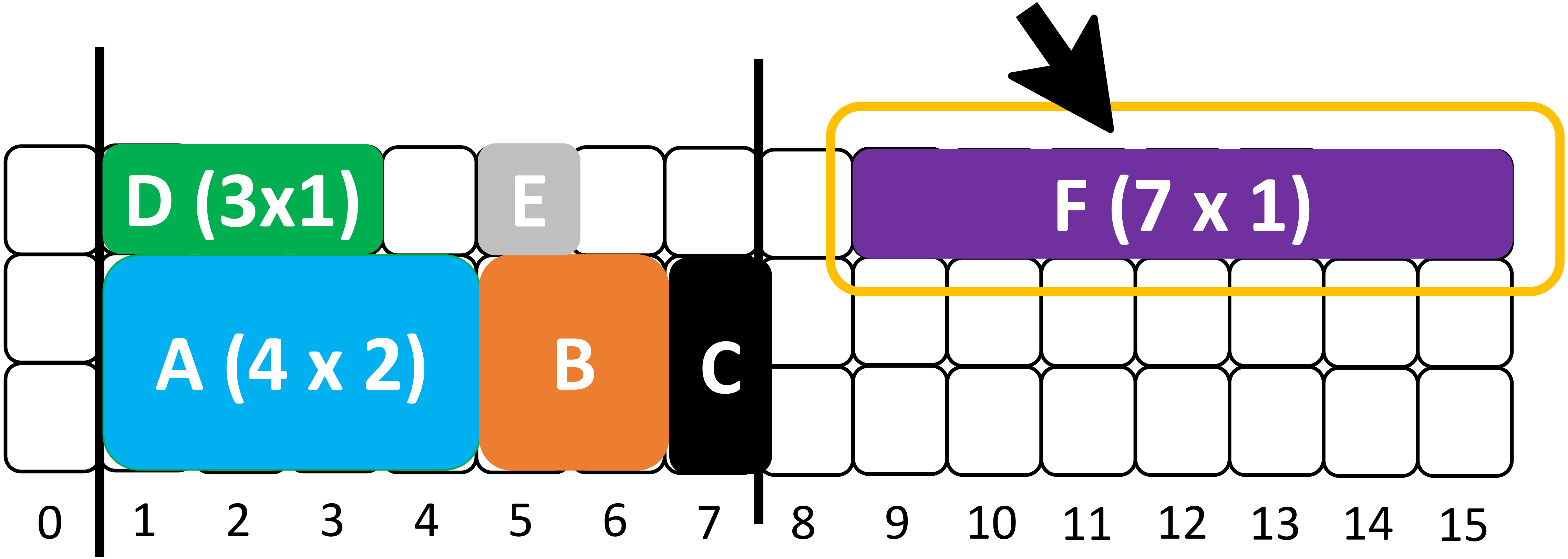
Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

These tasks must complete within this range 1-7



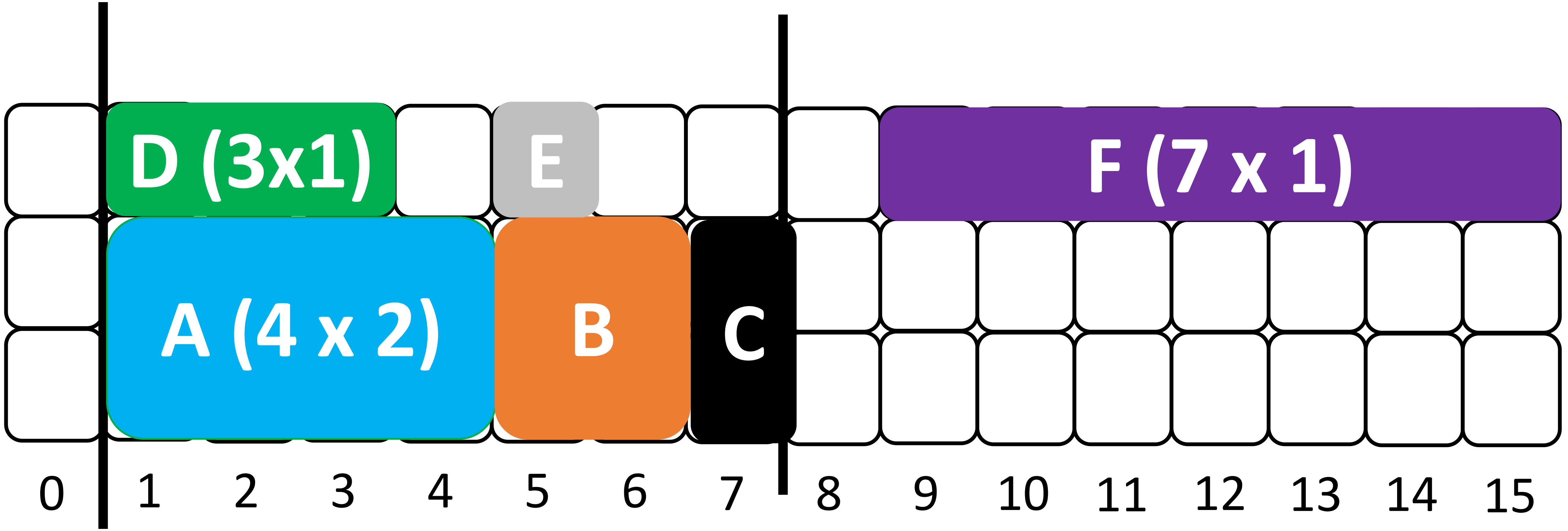
Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Task F can be completed any time

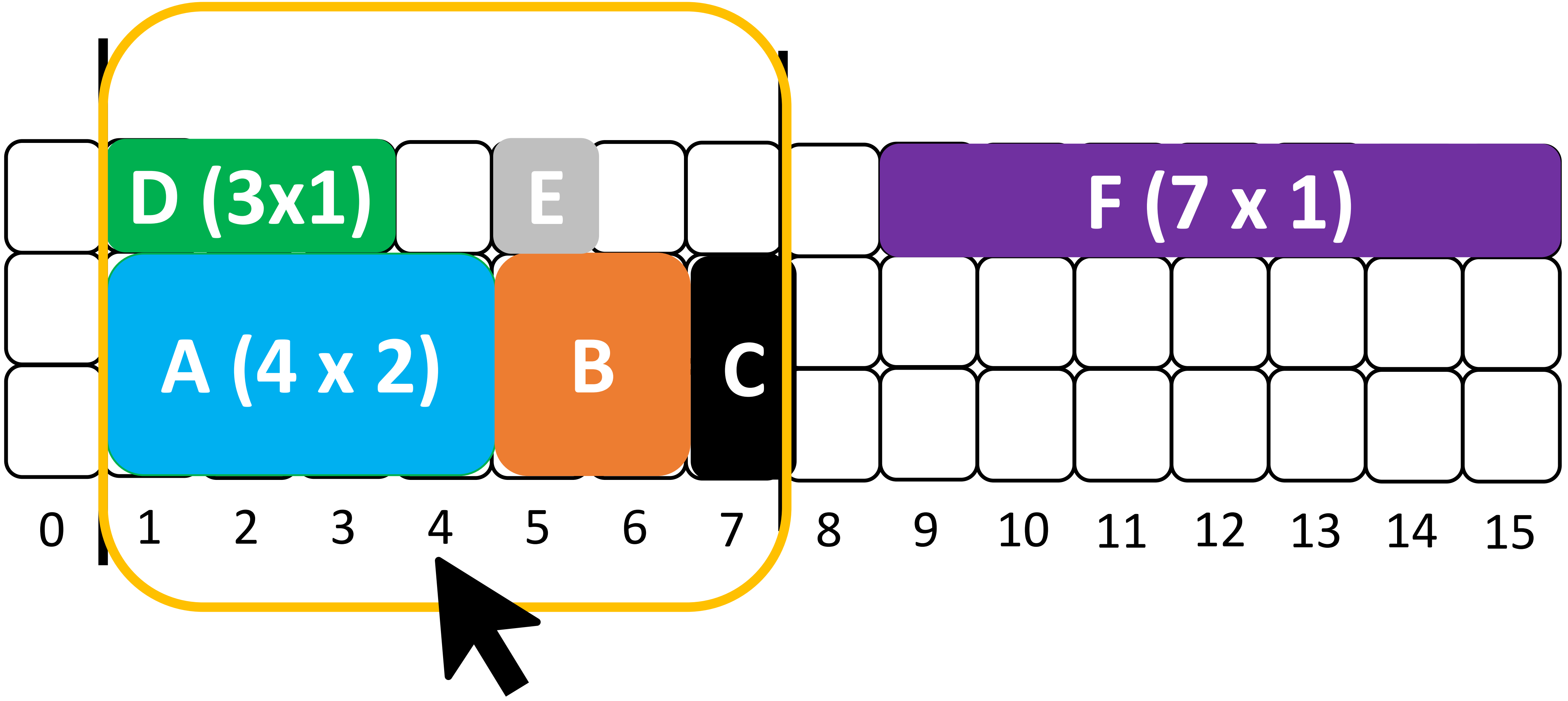


Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

What does timetabling reasoning infer?

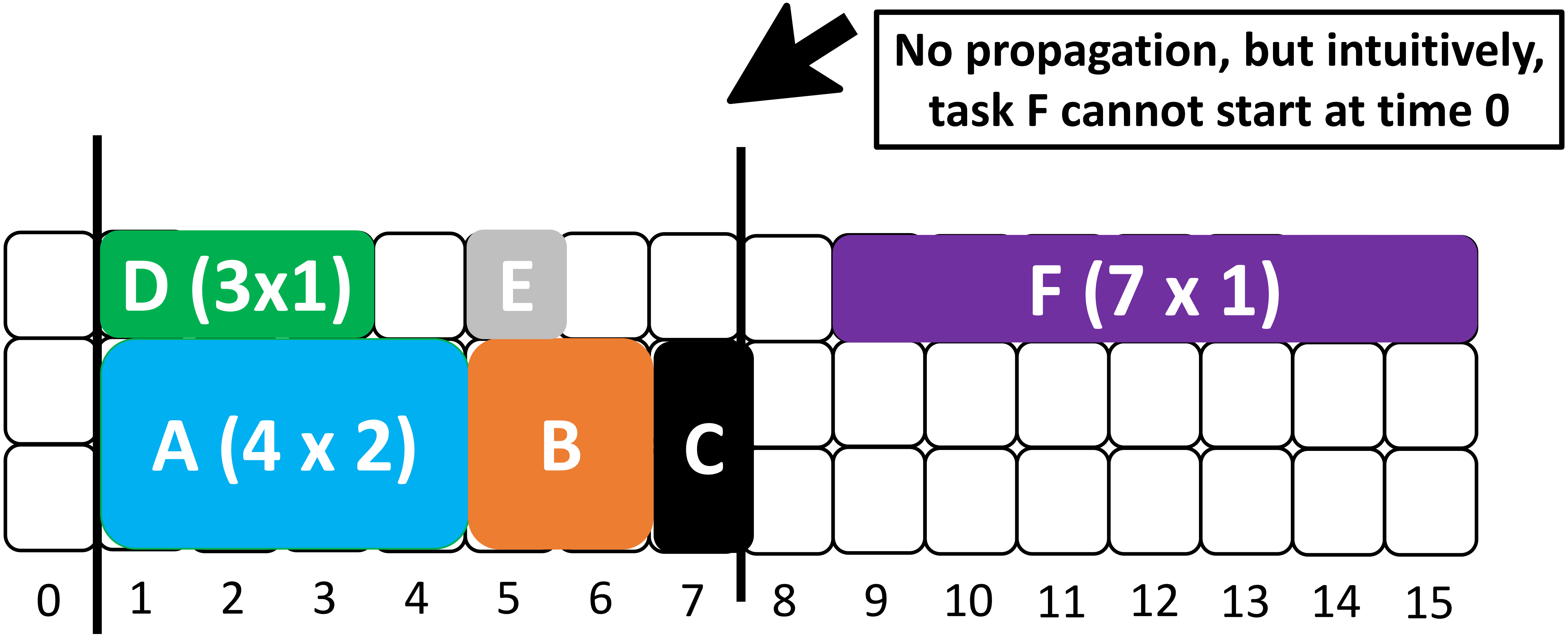


Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

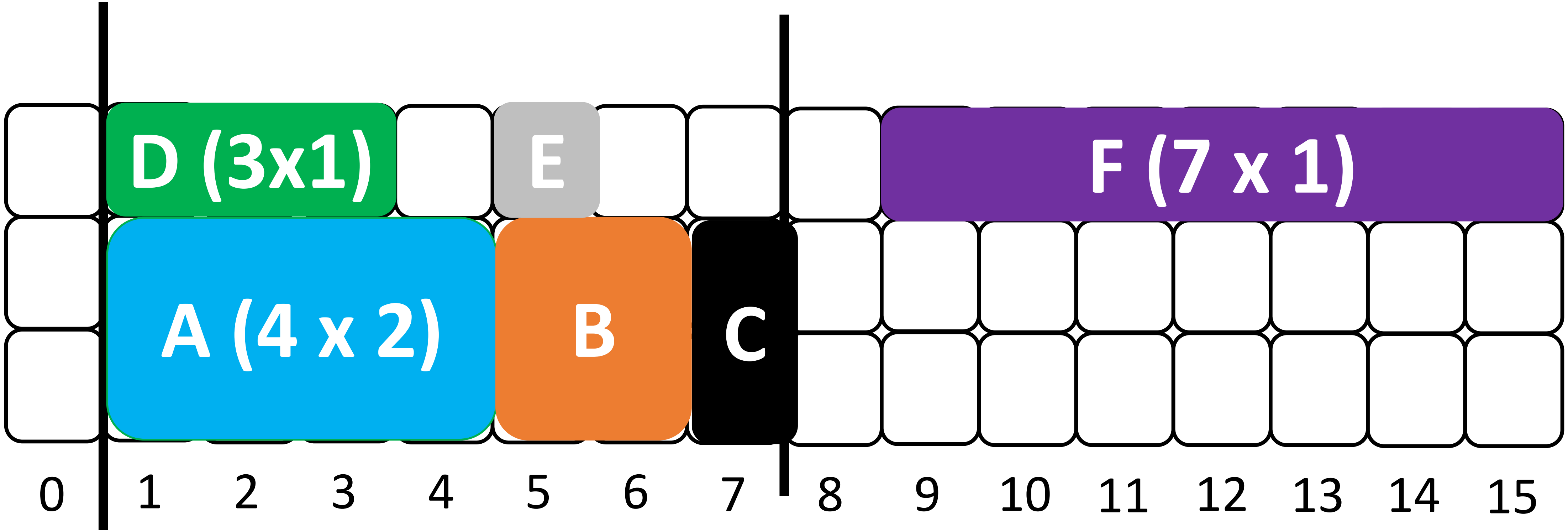


Only Task A has a mandatory part at time 4 → no propagation

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)



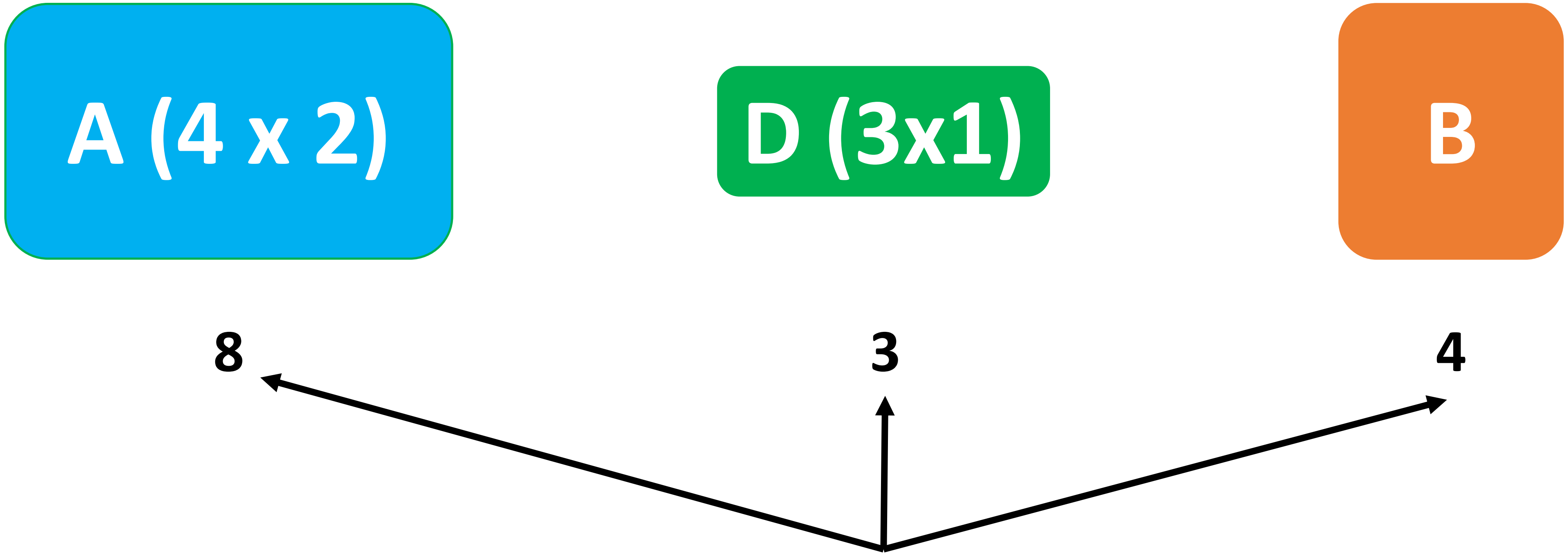
Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)



Need something different than timetabling!

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

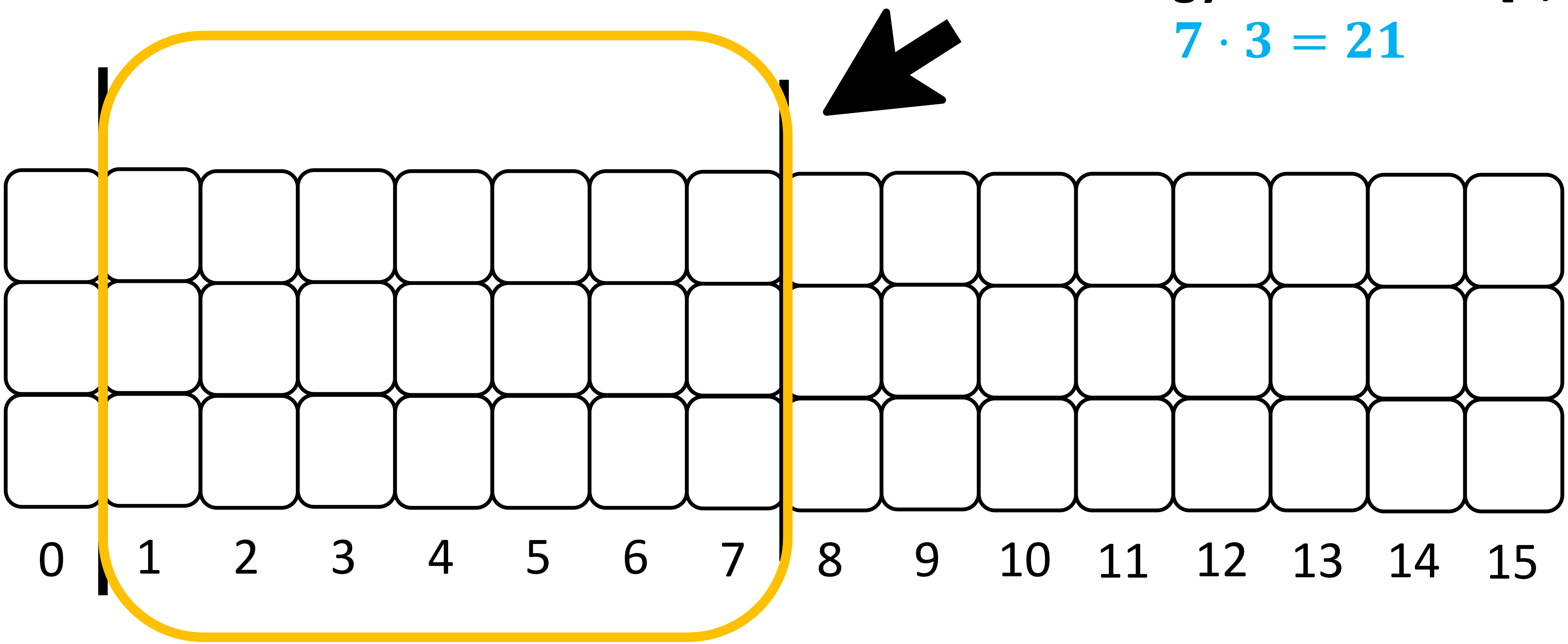
Energetic reasoning



$$\text{energy(task)} = \text{duration} \cdot \text{resource\_demand}$$

Total energy available in [1, 7]:

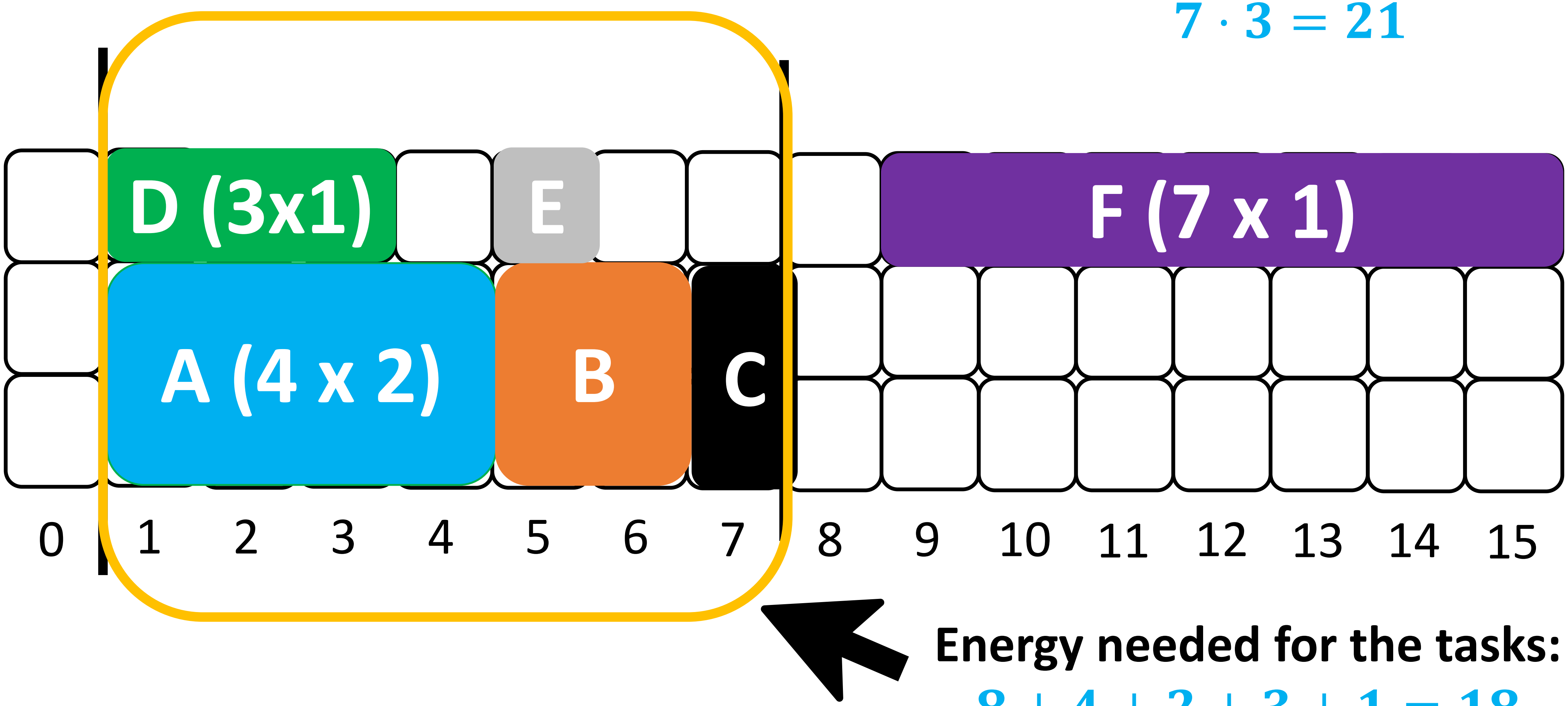
$7 \cdot 3 = 21$



Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Total energy available in [1, 7]:

$7 \cdot 3 = 21$

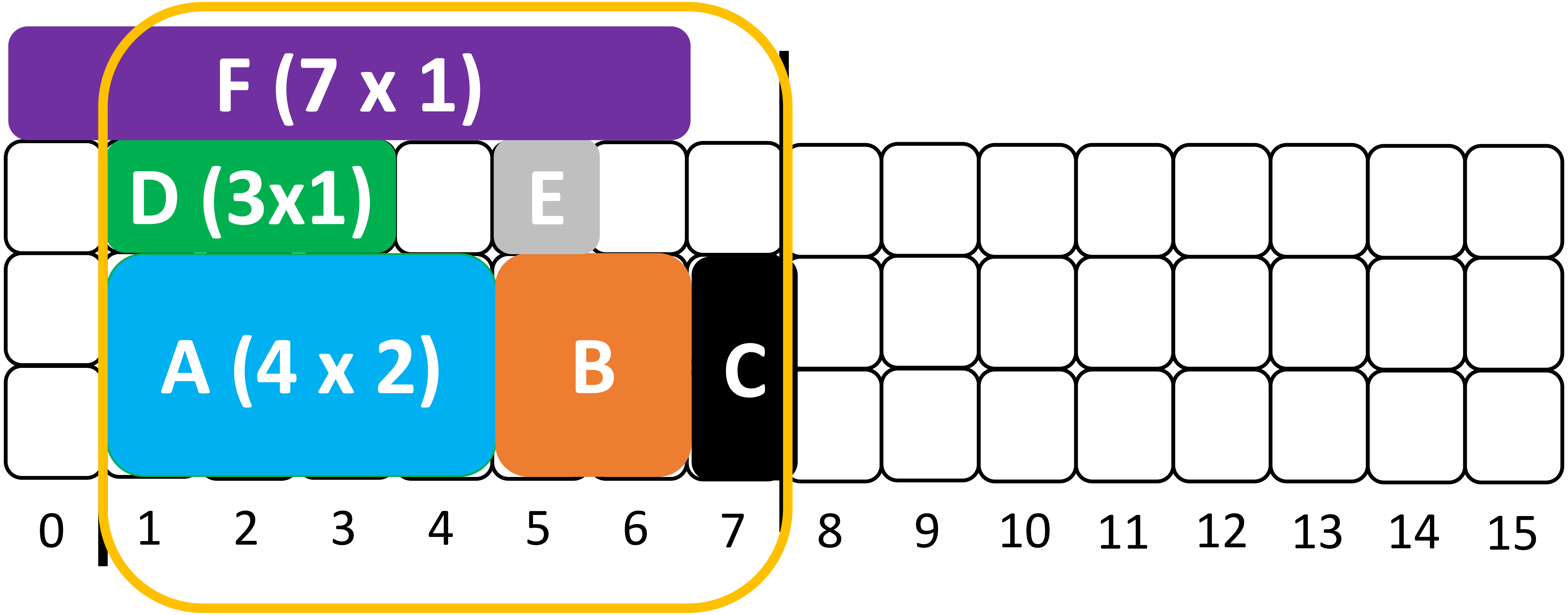


Energy needed for the tasks:

$8 + 4 + 2 + 3 + 1 = 18$

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

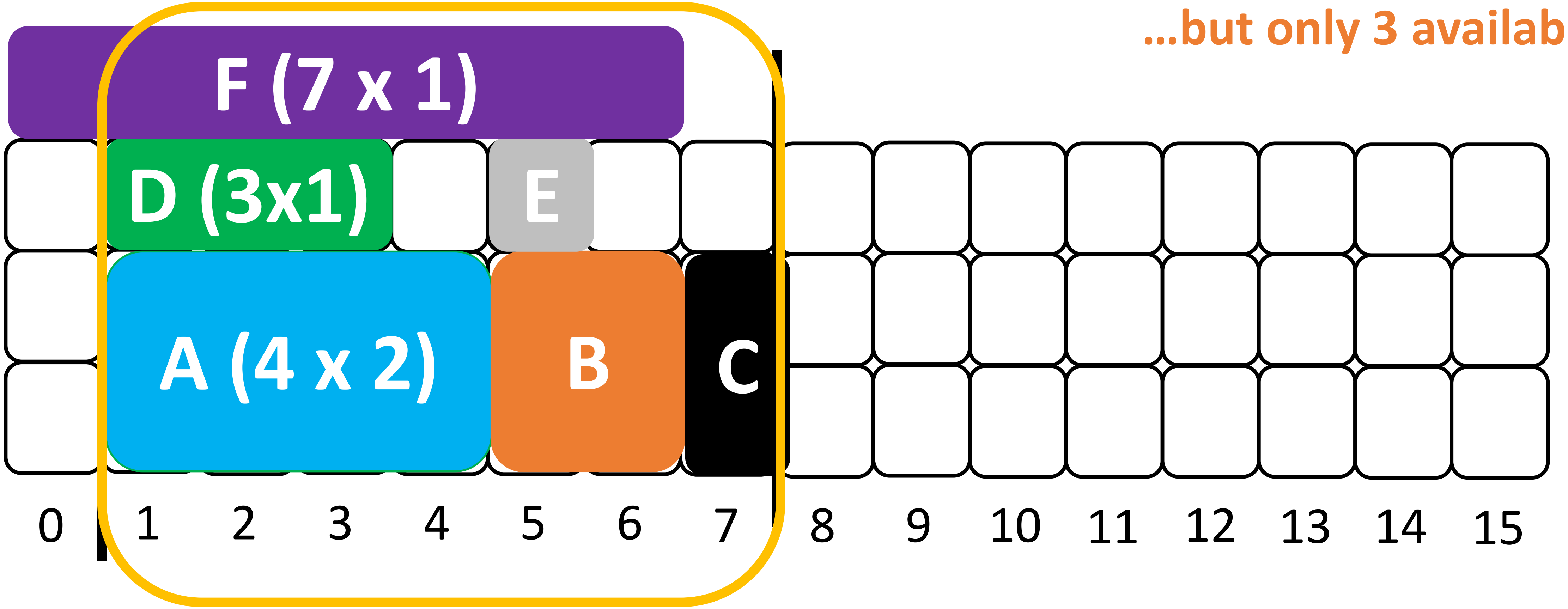
Place task F at its earliest starting time…
How much energy would task F need from [1, 7]?



Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Place task F at its earliest starting time…
How much energy would task F need from [1, 7]? ➔ 6

…but only 3 available



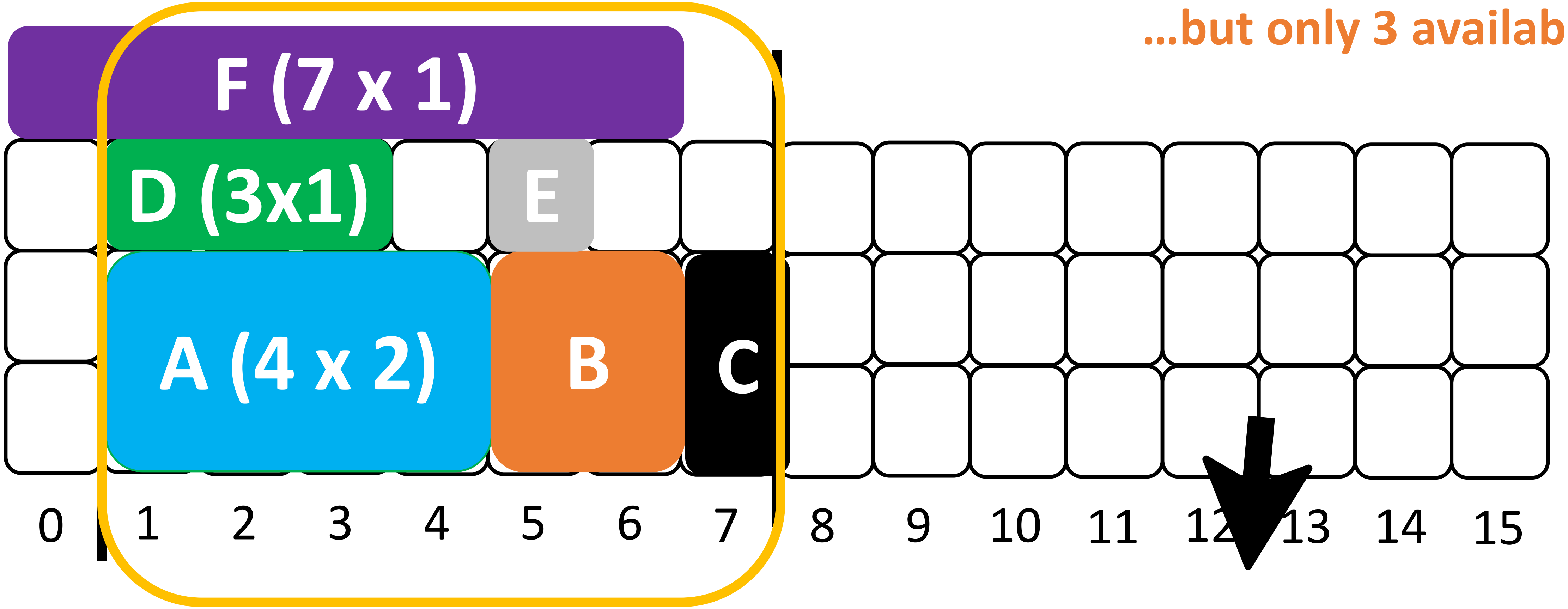
Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Place task F at its earliest starting time…

How much energy would task F need from [1, 7]?

6

…but only 3 available

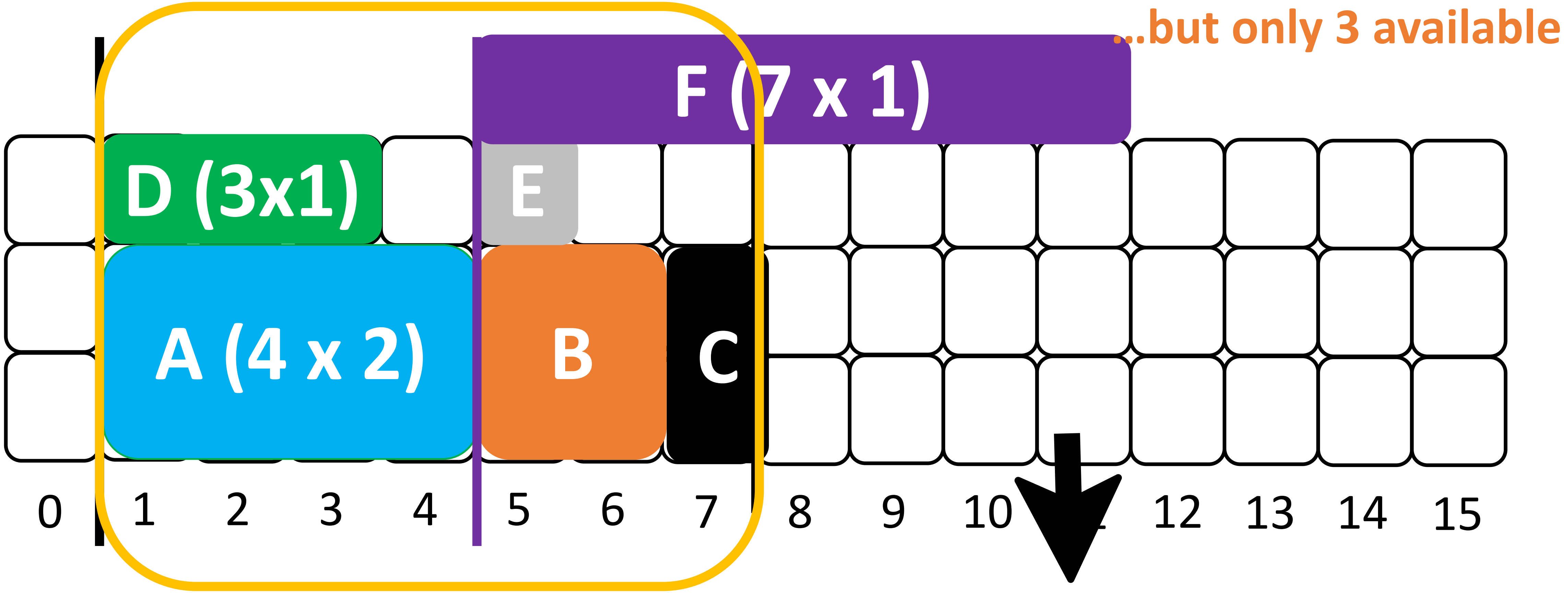


Remove time 0 from Task F!

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Place task F at its earliest starting time…
How much energy would task F need from [1, 7]?

6



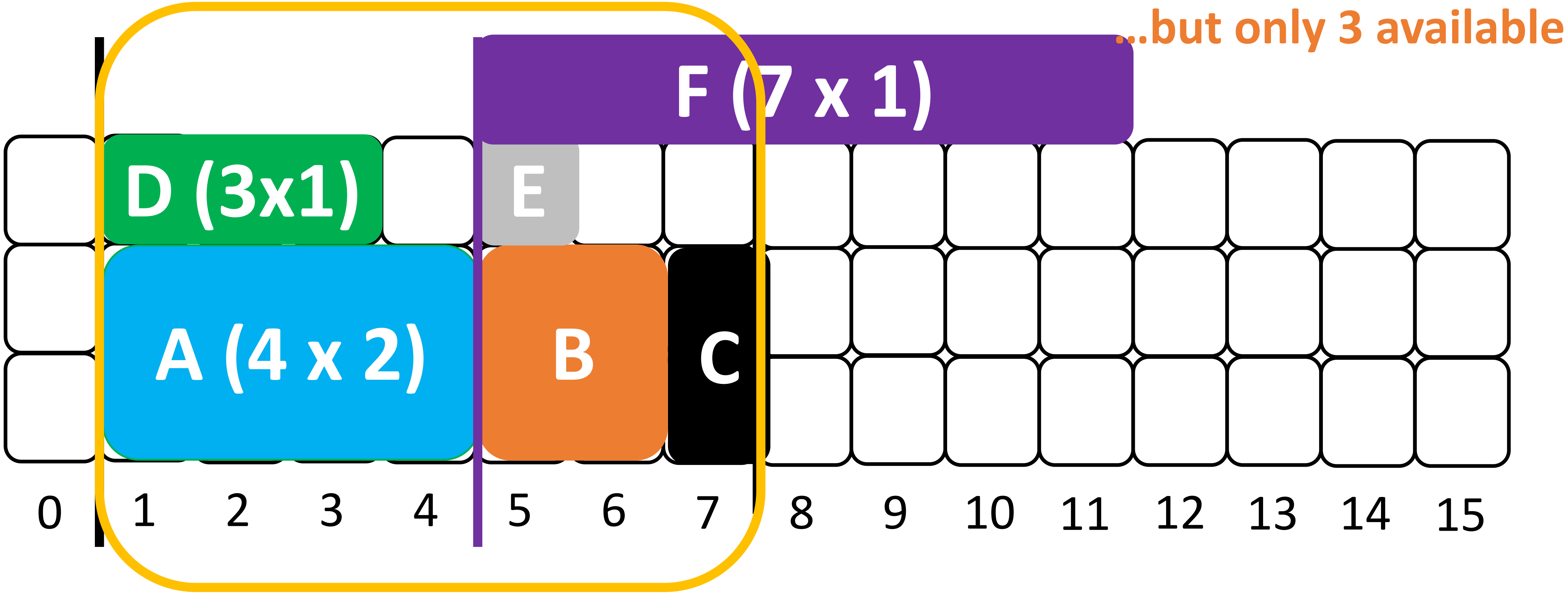
(only three energy left for task F, so it can start earliest at time 5)

Even better: $\mathrm{LB}(\text{Task F}) \geq 5$

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Place task F at its earliest starting time...
How much energy would task F need from [1, 7]?

6



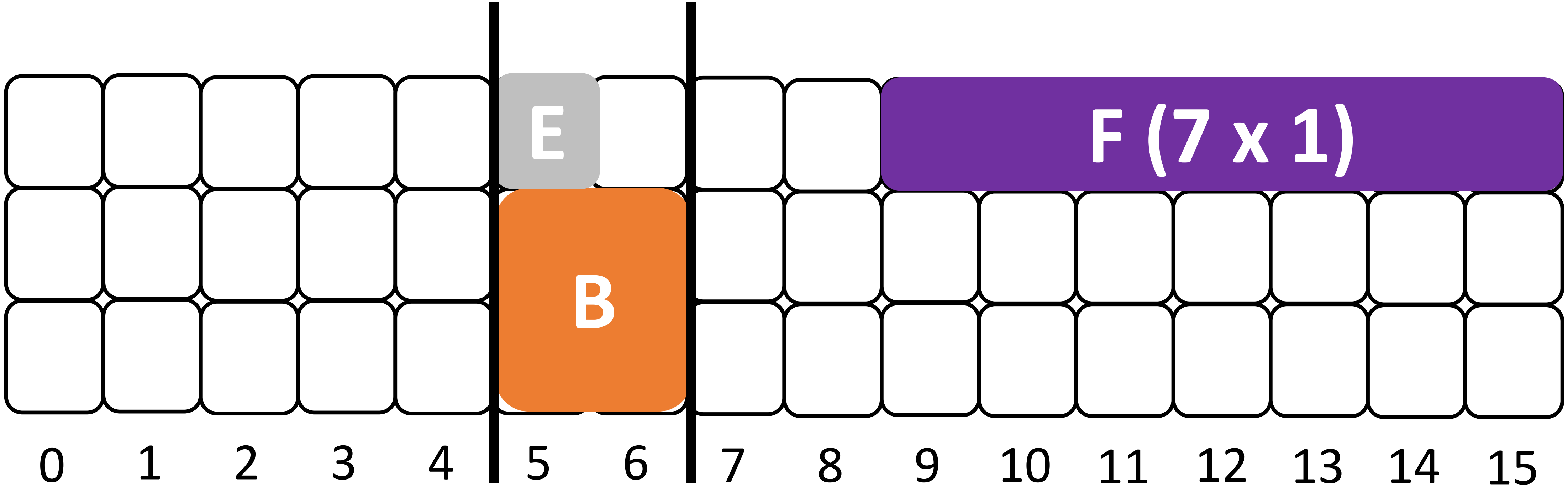
Better than timetabling (in this case!)

$\mathrm{LB}(\text{Task F}) \geq 5$

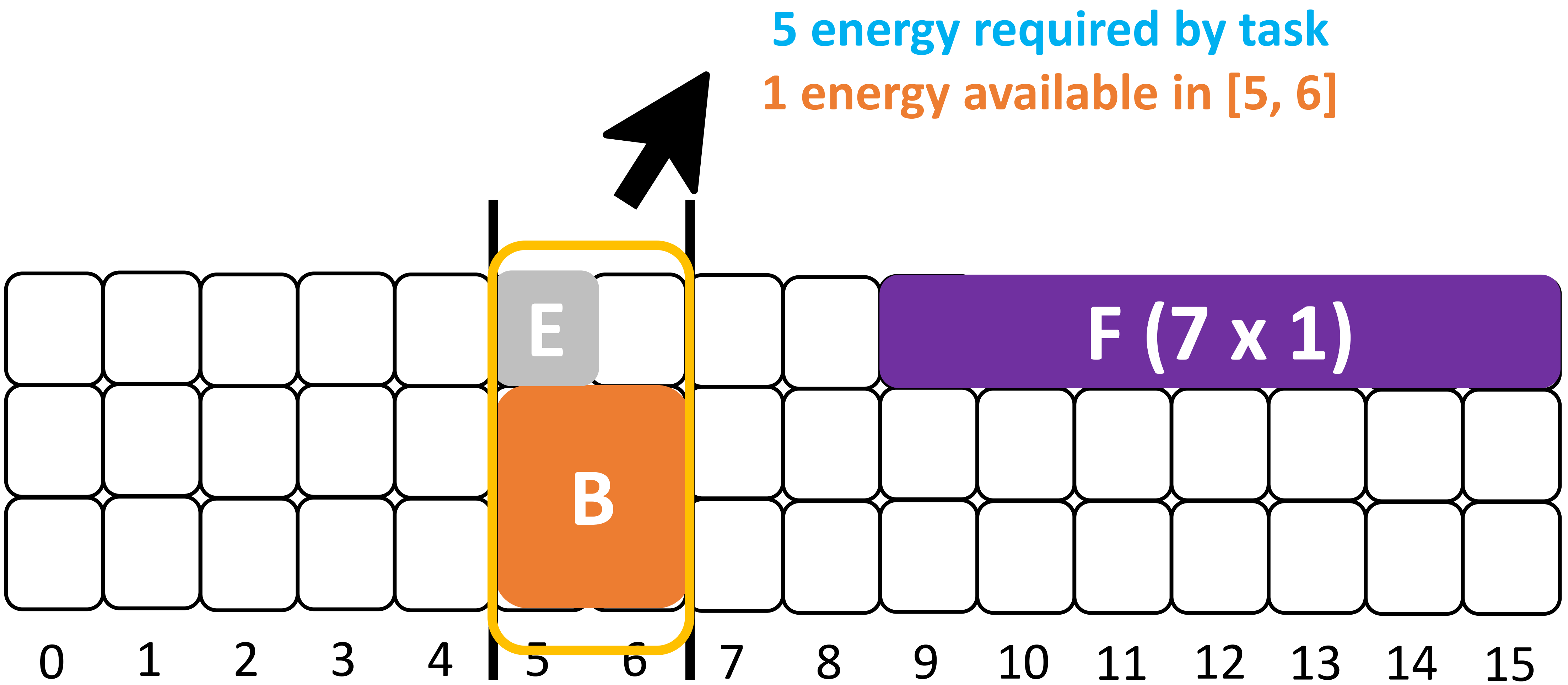
Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

New example

Timetabling does not lead to any propagation



Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

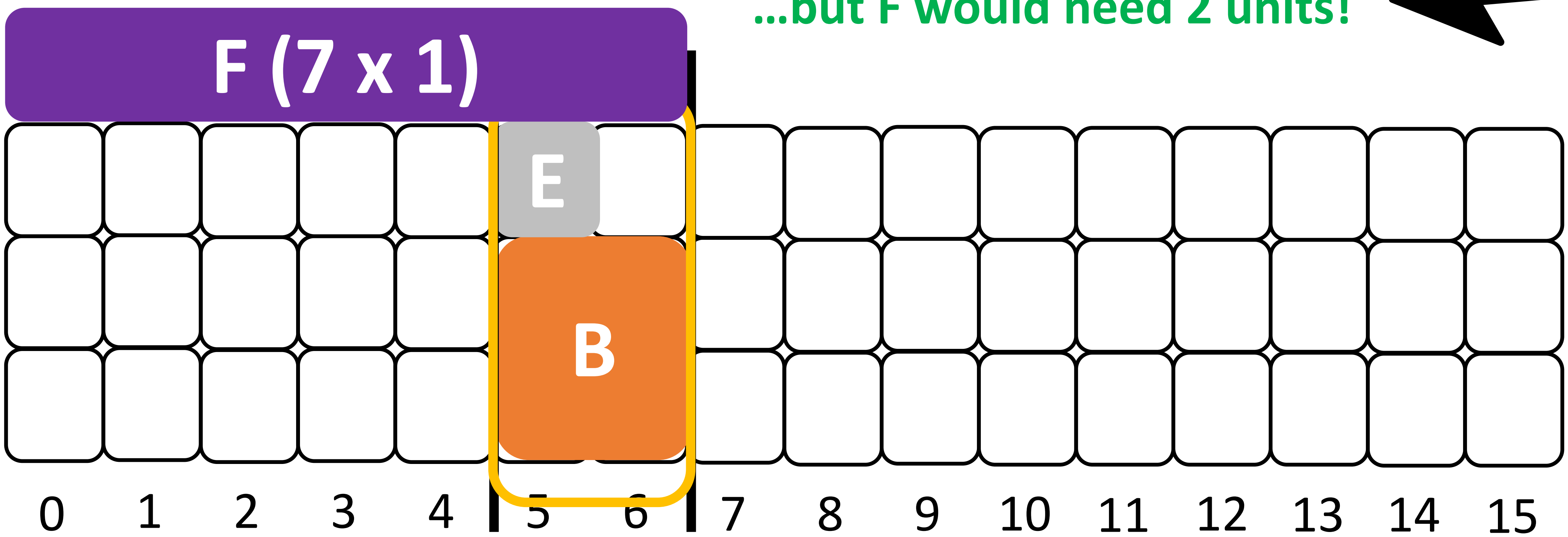


Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

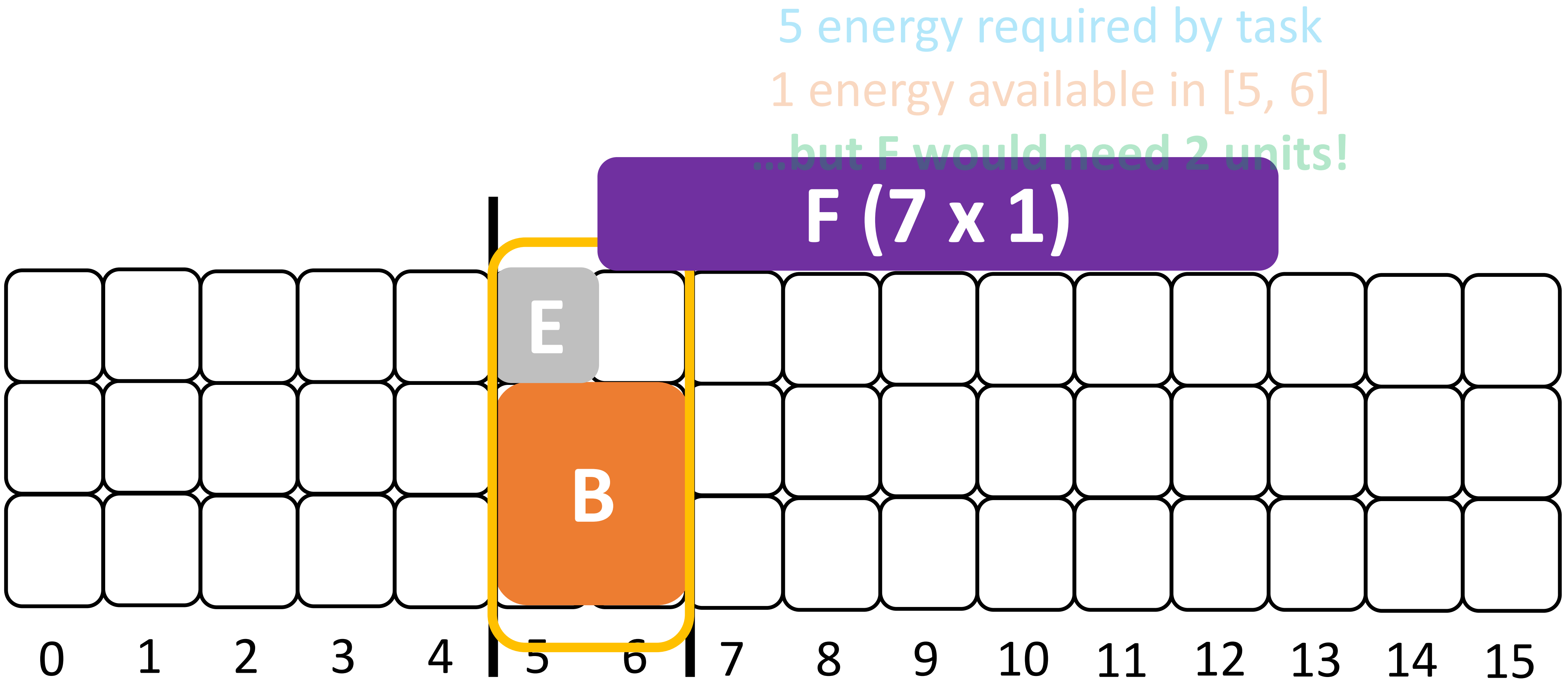
5 energy required by task

1 energy available in [5, 6]

...but F would need 2 units!



Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)



(only one energy left for task F, so it can start earliest at time 6)

$\mathbf{LB(Task\ F) \geq 6}$

Example adapted from A quadratic edge-finding filtering algorithm for cumulative resource constraints– Kameugne et al. (2013)

Energetic reasoning subsumes Timetabling!

But more expensive!

Summary

Cumulative Constraint

Timetable Reasoning

Energetic Reasoning

Summary

- All-Different
  - Decomposition
  - Hall Sets

- Cumulative Constraint
  - Timetable Reasoning
  - Energetic Reasoning

Checker → simple!

Propagation → Flow

Next time…

Conflict Analysis

Derive new constraints
by combining existing constraints

Backtrack more than one decision level